

Using combinatorics to understand Dyson-Schwinger equations

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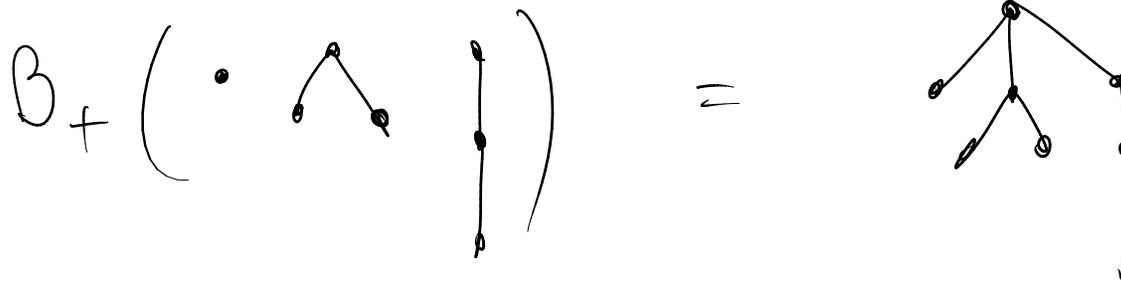
St. Johns

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Building trees

Let $B_+(F)$ be the tree constructed by adding a new root above each tree from the forest F .

Eg:



Tree Hopf algebra

There is a Hopf algebra structure on rooted trees. The coproduct is given by

$$\Delta(T) = \sum_c P_c(T) \otimes R_c(T)$$

and extended multiplicatively.

Eg:

The diagram shows the coproduct of a tree with a root node (blue) and two children (one red, one black). The tree is enclosed in large parentheses. The result is a sum of terms, each representing a tree with a circled node and a vertical line (representing the identity element $\mathbb{1}$). The terms are arranged in two rows. The first row contains: a tree with the left child circled and a vertical line to its right; a tree with the right child circled and a vertical line to its left; a tree with the root circled and a vertical line below it; and a tree with the root circled and a vertical line above it. The second row contains: a tree with the left child circled and a vertical line below it; and a tree with the right child circled and a vertical line above it. Ellipses are used to indicate that there are more terms in the sum.

The product is disjoint union. This is the Connes-Kreimer Hopf algebra.

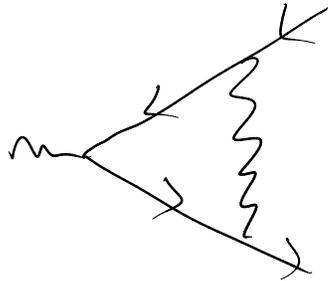
Feynman graphs

Feynman graphs describe interactions in particle physics. They are graphs built of half-edges with specified

- edge types (oriented and unoriented) and
- vertex types

They may have *external edges*.

Eg: QED



A Feynman graph is 1PI if it is 2-edge-connected.

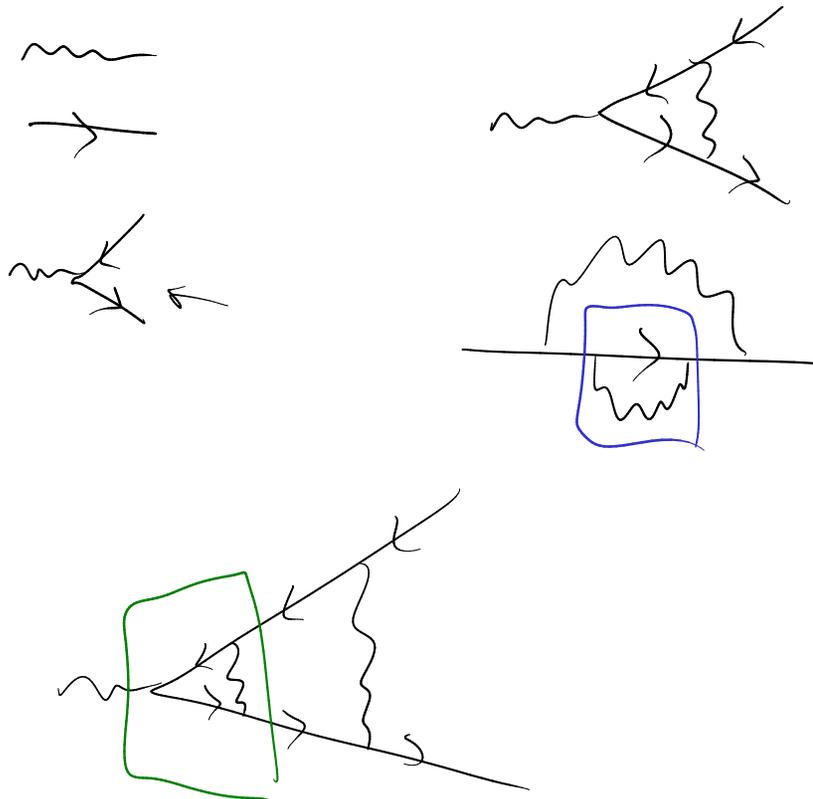
Feynman rules map Feynman graphs to (formal) integrals.

Divergences

A Feynman graph is *divergent* if the associated integral diverges. If we have set up our types correctly, this will occur when the external edges of the graph give one of the edge or vertex types.

Eg:

QED



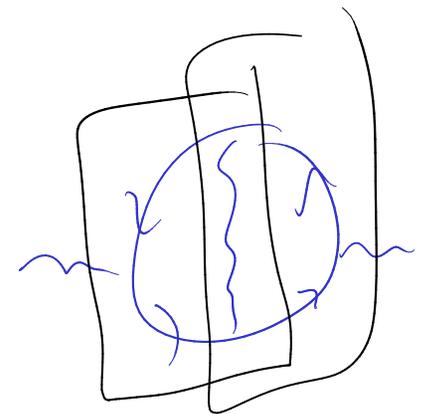
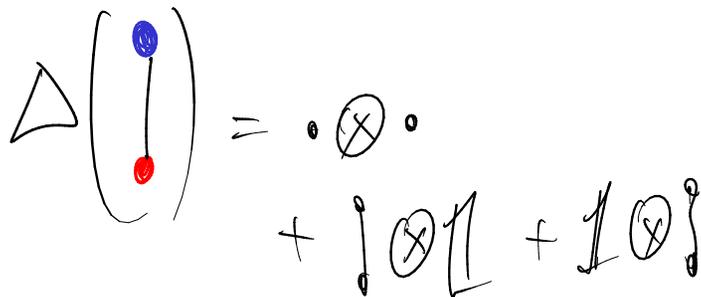
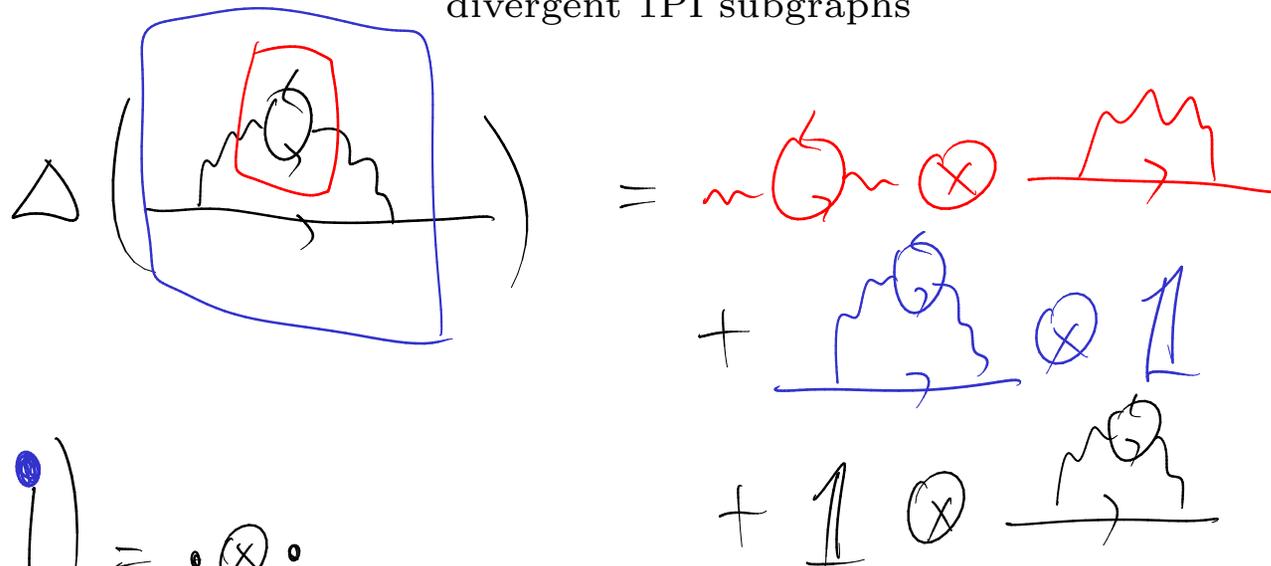
Renormalization Hopf algebras

There are corresponding Hopf algebras of Feynman graphs

$$\Delta(G) = \sum_{\gamma \text{ product of divergent 1PI subgraphs}} \gamma \otimes G / \gamma$$

pruned part
root part

Eg:



A graph is *primitive* if it has no divergent subgraphs.

B_+ for graphs

Write B_+^γ for insertion into the primitive graph γ .

Eg:

The diagram shows the operation B_+ applied to a graph. On the left, the symbol B_+ is followed by a horizontal line with a jagged, crown-like shape above it. This is followed by a large pair of parentheses containing a similar horizontal line with a jagged shape above it. An equals sign follows, and then a single horizontal line with a jagged shape above it that is the result of inserting the first jagged shape into the second.

By weighting the insertions by an appropriate combinatorial coefficient, and, where necessary, working in a quotient algebra (Ward identities...) we obtain that B_+ is a Hochschild 1-cocycle for the renormalization Hopf algebra.

$$\Delta B_+ = (\text{id} \otimes B_+) \Delta + B_+ \otimes \mathbb{I}$$

1-cocycle property for rooted trees

Let's check the 1-cocycle property for rooted trees. We have

$$\Delta(T) = \sum_c P_c(T) \otimes R_c(T)$$

We want

$$\Delta B_+ = (\text{id} \otimes B_+) \Delta + B_+ \otimes \mathbb{I}$$

This is an important property. Why? Where else does this come up?

Combinatorial Dyson-Schwinger equations

The recurrences in Feynman diagrams which describe how to build the graphs of a theory out of smaller graphs are the *combinatorial Dyson-Schwinger equations*.

They are a kind of combinatorial specification language for trees and related structures.

Eg:

$$X = \mathbb{I} + xB_+(X)$$

$$X = \begin{array}{c} \diagup \\ \diagdown \end{array} + x \bullet + x^2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + x^3 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + x^4 \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} + \dots$$

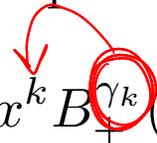
Eg:

$$X = \mathbb{I} - xB_+ \left(\frac{1}{X} \right)$$

$$X = \begin{array}{c} \text{I} \\ \text{---} \\ \text{I} \end{array} - x \bullet - x^2 \downarrow - x^3 \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \right) + \dots$$

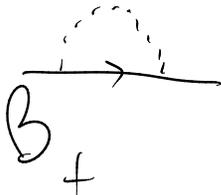
A particular form

The same idea holds for Feynman graphs. For today

$$X = \mathbb{I} \pm \sum_{k \geq 1} x^k B_{\mp}^{\gamma_k}(XQ^k)$$


where $Q = X^{-s}$.

Eg (Broadhurst and Kreimer):

$$X = \mathbb{1} - x B_{+}^{\text{loop}} \left(\frac{1}{X} \right)$$


$$X = \mathbb{1} - x B_{+}^{\text{loop}} - x^2 B_{+}^{\text{bubble}} - \dots$$

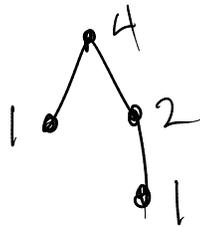

Analytic Dyson-Schwinger equations

Recall Feynman rules take graph to (formal) integrals. Renormalization tells us how to get convergent results.

Analytic Dyson-Schwinger equations are the result of applying Feynman rules to combinatorial Dyson-Schwinger equations.

Toy Feynman rules remain combinatorial on the “analytic” side.

Eg:



$$4 \cdot 2 \cdot 1 \cdot 1$$



More generally

More generally the analytic DSEs are integral equations with many parameters.

- The recursive structure of the DSE takes care of the recursive structure of renormalization.
- The counting variable x becomes the coupling constant
- We get new analytic variables coming from the external momenta. For today just one variable L .
- X becomes the Green function $G(x, L)$.

Continuing the example

In the Broadhurst-Kreimer Yukawa example

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2) (k+q)^2} - \dots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.

This has the same recursive structure as $X = \mathbb{I} - x B_+ \left(\frac{1}{X} \right)$

Manipulate

Now

- plug in $G(x, L) = 1 - \sum \gamma_k(x) L^k$
- use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
- switch the order of \int and ∂

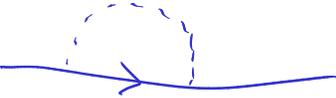
to obtain

$$G(x, L) = 1 \pm \sum_{k \geq 1} x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$

Where $F_k(\rho)$ is the integral for γ_k regularized by a parameter ρ which marks the insertion place.

Reductions

This example works so well because our Dyson-Schwinger equation had

- One primitive graph 
- which had a particularly nice integral (scaled just a geometric series)
- inserted into one place

The program of arXiv:0810.2249, *Memoir. Am. Math. Soc.* 211, no. 995, with an important improvement in arXiv:1302.0080, was to generalize this nice situation into a general reduction process for Dyson-Schwinger equations.

Some steps make combinatorial sense, others do not.

A sequence

$$r_1 = f_0$$

$$r_2 = f_0 f_1 - f_0^2$$

$$r_3 = -4f_0^2 f_1 + 3f_2 f_0^2 + f_0 f_1^2$$

$$r_4 = 11f_2 f_0^2 f_1 - 9f_0^2 f_1^2 - 18f_2 f_0^3 + f_0 f_1^3 + 15f_3 f_0^3$$

$$r_5 = 86f_3 f_0^3 f_1 - 120f_3 f_0^4 - 16f_0^2 f_1^3 + f_0 f_1^4 + 30f_2^2 f_0^3 + 105f_0^4 f_4 \\ - 112f_2 f_0^3 f_1 + 26f_2 f_0^2 f_1^2$$

How far can we go combinatorially

Keep the top and bottom points and generalize the middle one. Set $s = 2$ and $k = 1$, that is,

$$G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \dots$$

Write

$$G(x, L) = 1 - \sum_{n \geq 1} \gamma_n(x) L^n$$

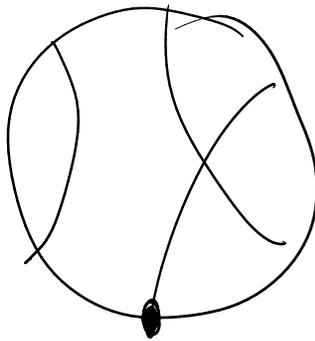
then the Dyson-Schwinger equation determines the γ_n in terms of the f_i , *but not in a nice way.*

Rooted connected chord diagrams

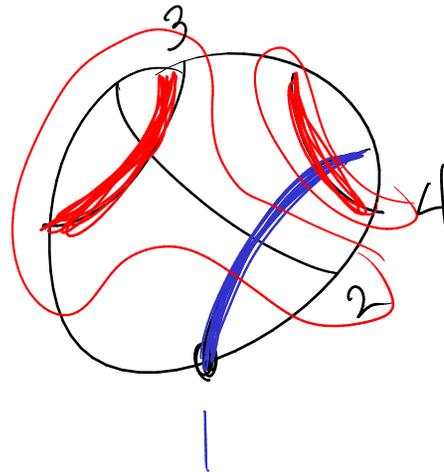
A chord diagram is *rooted* if it has a distinguished vertex.

A chord diagram is *connected* if no set of chords can be separated from the others by a line.

Eg:



not
connected



These are really just irreducible matchings of points along a line.

Intersection graphs and bad chords

The *intersection graph* of a chord diagram is the graph with

- **vertices:** the chords of the diagram
- **adjacencies:** vertices where the corresponding chords cross.

The root and counterclockwise order of the chord diagram let us direct the intersection graph.

Say a chord is *terminal* if it is terminal in the directed intersection graph.

Eg:

Recursive chord order

Let C be a connected rooted chord diagram. Order the chords recursively:

- c_1 is the root chord
- Order the connected components of $C \setminus c_1$ as they first appear running counterclockwise, D_1, D_2, \dots . Recursively order the chords of D_1 , then of D_2 , and so on.

Eg:

terminal chords are the base case
 \rightsquigarrow only one chord in
a component

The terminal chords come from applications of the base case: a diagram with only one chord.

Index lists

Let C be a connected rooted chord diagram. Define

- $w(C) = \{i : c_i \text{ is terminal}\}$ (using the recursive chord order)
- $i(C)$ is the list of differences of successive elements in $w(C)$ padded with 0s to contain $|C| - 1$ elements.
- $b(C)$ is the minimum index of a terminal chord.

Eg:

These will be our index lists: $f_C = \prod_{i \in i(C)} f_i$.

Result

Theorem 1

$$\gamma_i(x) = \frac{(-1)^i}{i!} \sum_{\substack{C \\ b(C) \geq i}} x^{|C|} f_{i(C)} f_{b(C)-i-1}$$

where C runs over rooted chord diagrams, solves the DSE

$$G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \dots$$
$$G(x, L) = 1 - \sum_{n \geq 1} \gamma_n(x) L^n$$

The proof is by manipulating two recurrences.

Conclusions

This is a new and unexpected expansion. It gives the Green function as a kind of multivariate generating function over chord diagrams.

[arXiv:1210.5457](https://arxiv.org/abs/1210.5457) with Nicolas Marie.

Now I'm looking at two primitives with Markus Hihn. Other future plans include other values of s .

It's still an expansion – how to resum?