The asymptotic value of the independence ratio for the direct graph power

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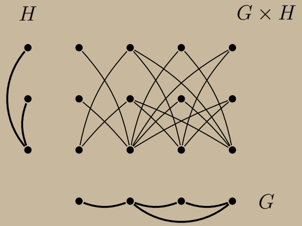
independence ratio of a graph G: $i(G) = \frac{\alpha(G)}{|V(G)|}$

direct product of two graphs G and H:
the graph $G \times H$ for which
$V(G \times H) = V(G) \times V(H)$, and
$\{(x_1, y_1), (x_2, y_2)\} \in E(G \times H)$, iff
$\{x_1, x_2\} \in E(G)$ and $\{y_1, y_2\} \in E(H)$.



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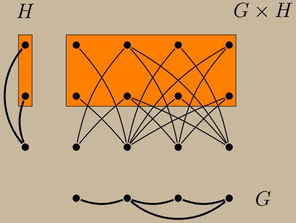

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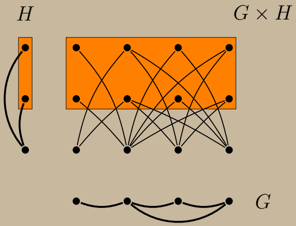

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Definition (Brown, Nowakowski, Rall - 1996.):
The asymptotic value of the independence ratio for the direct graph power is defined as

$$A(G) = \lim_{k \to \infty} i(G^{\times k}).$$

Results of Brown, Nowakowski and Rall

$$0 < i(G) \leq i(G^{\times 2}) \leq i(G^{\times 3}) \leq \cdots \leq A(G) \leq 1$$

Theorem (Brown, Nowakowski, Rall - 1996.):
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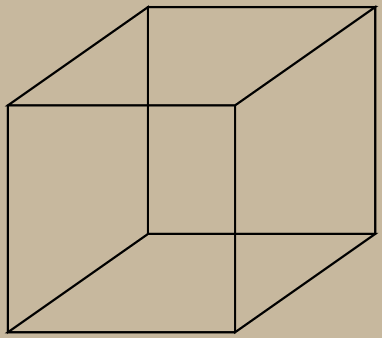

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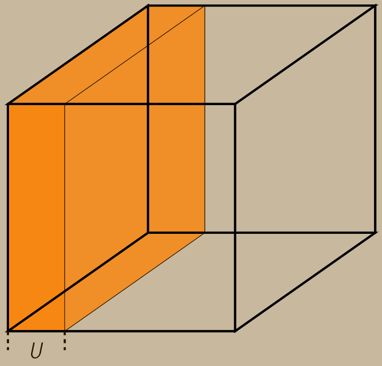

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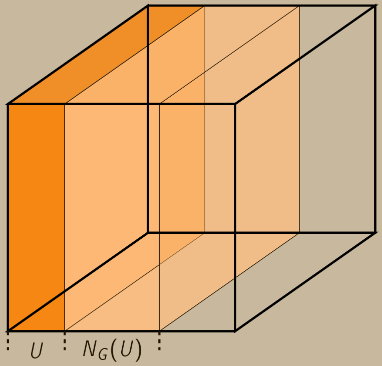

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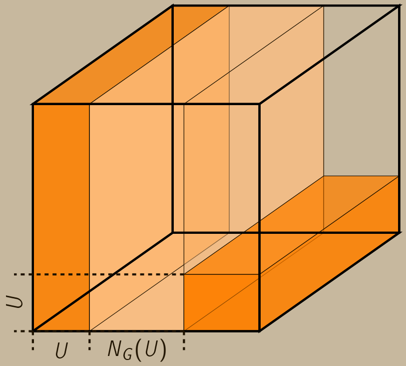

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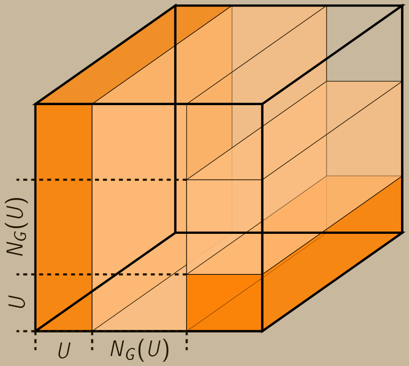

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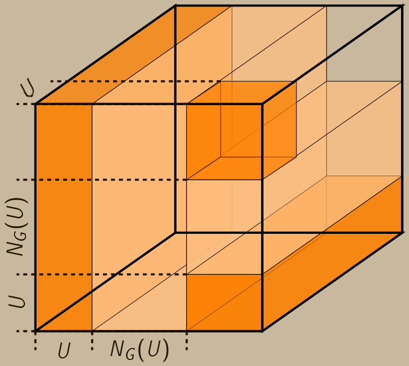

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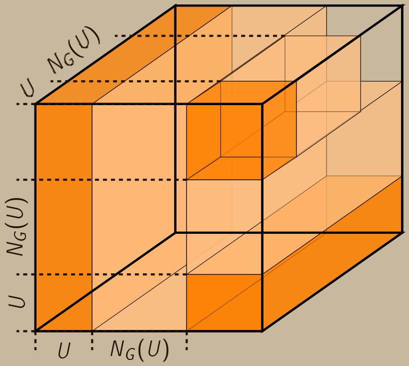

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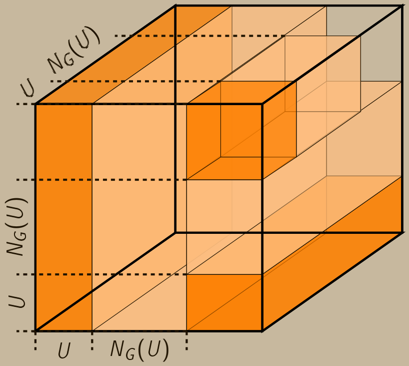

there exists an independent set U_k of $G^{\times k}$ such that

$$\frac{|U_k|}{|U_k|+|N_{G^{\times k}}(U_k)|} \geq \frac{|U|}{|U|+|N_G(U)|}$$

and

$$\lim_{k\to\infty} \frac{|U_k|}{|V(G^{\times k})|} = \frac{|U|}{|U|+|N_G(U)|}$$

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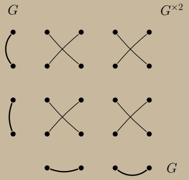
if $\alpha(G) > \frac{1}{2}|V(G)|$ then $A(G) = 1$
if $\alpha(G) = \frac{1}{2}|V(G)|$ then
G has a perfect matching,
therefore $G^{\times k}$ also has one ($\forall k$)
and $i(G^{\times k}) \leq \frac{1}{2}$ thus $A(G) = \frac{1}{2}$



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Observation (Alon, Lubetzky): $A(G) \geq i^*_{max}(G)$, where

$$i_{max}(G) = \max_{U \text{ independent in } G} \frac{|U|}{|U|+|N_G(U)|}$$

$$i^*_{max}(G) = \begin{cases} i_{max}(G), & \text{if } i_{max}(G) \leq \frac{1}{2} \\ 1, & \text{if } i_{max}(G) > \frac{1}{2} \end{cases}.$$

Questions of Alon and Lubetzky

$$i(G) \overset{\exists G:<}{\leq} i_{max}(G) \overset{\exists G:<}{\leq} i^*_{max}(G) \leq A(G)$$

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Does every graph G satisfy $A(G) = i^*_{max}(G)$?

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It easily follows from the inequality

$$i^*_{max}(G \times H) \leq \max\{i^*_{max}(G), i^*_{max}(H)\}.$$

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Proposition (weaker inequality): $i(G \times H) \leq \max\{i^*_{max}(G), i^*_{max}(H)\}$

Consequences

Conjecture (BNR): $A(G \cup H) = \max\{A(G), A(H)\}$, where $A \cup G$ denotes the disjoint union of G and H.

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For any rational $r \in (0, \frac{1}{2}] \cup \{1\}$ there exists a graph G with $A(G) = r$.
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From $A(G) = i^*_{max}(G)$ **we obtain that**:

$A(G \cup H) = \max\{A(G), A(H)\}$.

$A(G)$ cannot be irrational.

Algorithmic aspects

Question (BNR): Is $A(G)$ computable?
And if so, what is its complexity?

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From $A(G) = i^*_{max}(G)$ **we also obtain that**:
The problem of deciding whether $A(G) > t$ for a given graph G and a value t, is NP-complete.

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For every graph G and H we have

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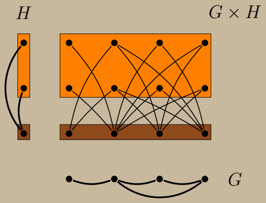

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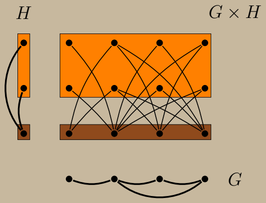

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The fractional version of the conjecture:
(χ_f denotes the fractional chromatic number of the graph.)

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}.$$

$\chi_f(G \times H) \leq \min\{\chi_f(G), \chi_f(H)\}$ is easy.
Tardif, 2005.: $\chi_f(G \times H) \geq \frac{1}{4}\min\{\chi_f(G), \chi_f(H)\}$.

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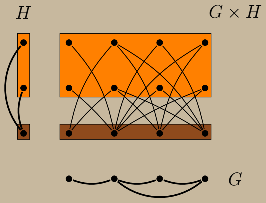

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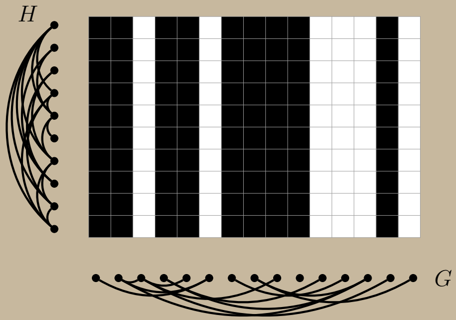
Theorem (Zhu - 2010.):
The fractional version of Hedetniemi's conjecture is true.
Corollary: The Burr-Erdős-Lovász conjecture is true.

The idea of the proof - Zhu's lemma



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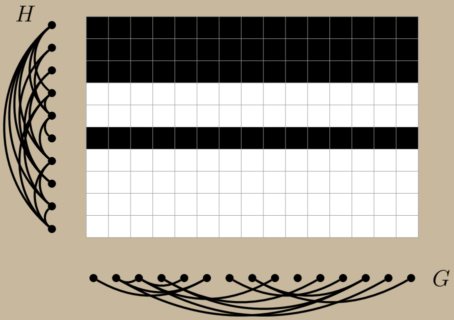

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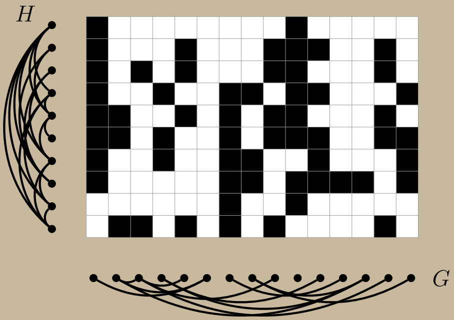

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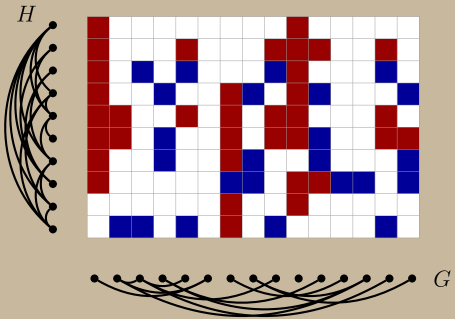

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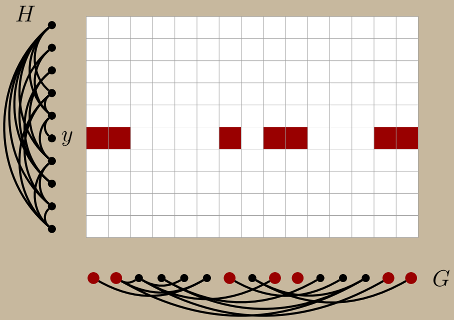

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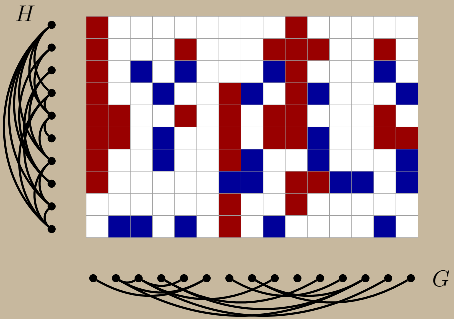

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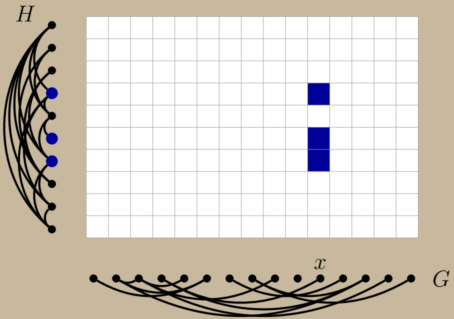

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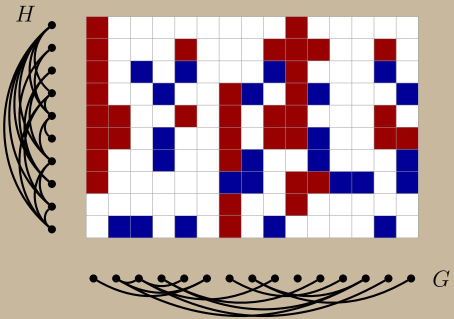

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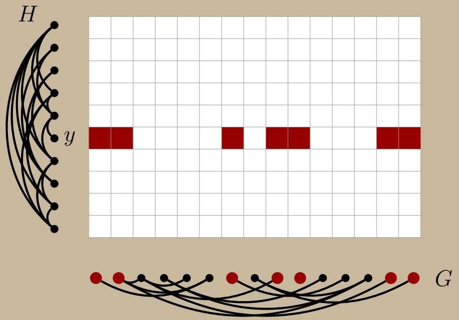

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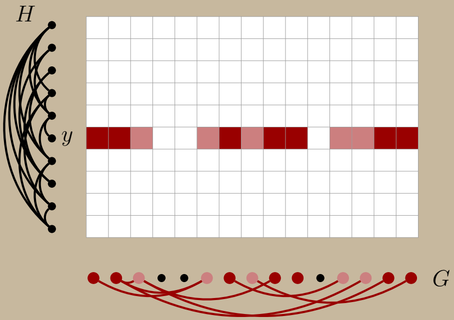

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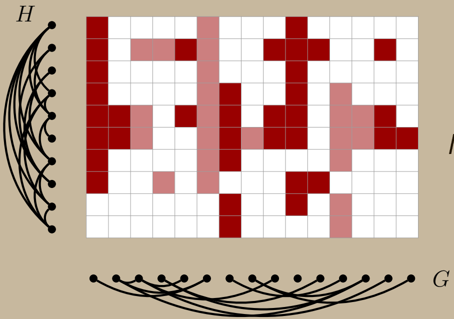

$$MA = \{(x, y) \in V(G \times H) : \exists (x', y) \in A, \{x, x'\} \in E(G)\}$$

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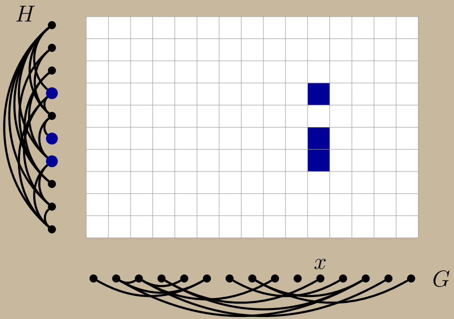

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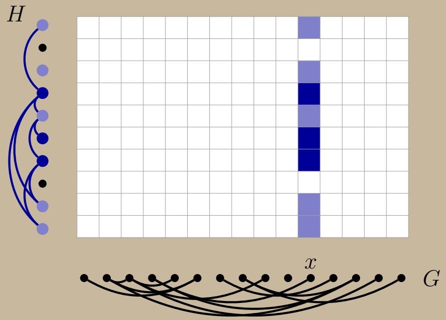

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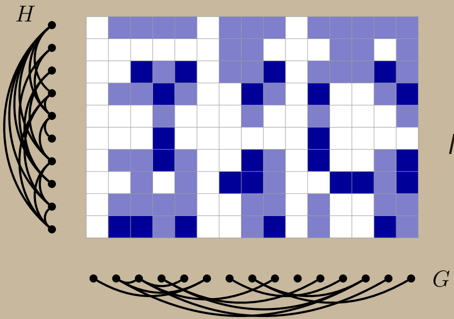

$$MB = \{(x,y) \in V(G \times H) : \exists (x,y') \in B, \{y,y'\} \in E(H)\}$$

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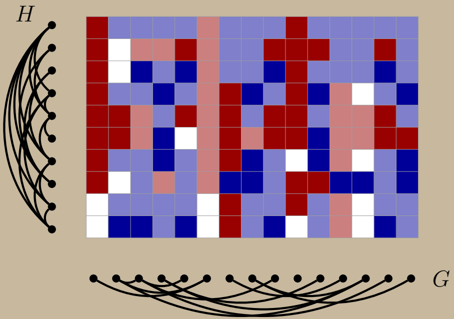

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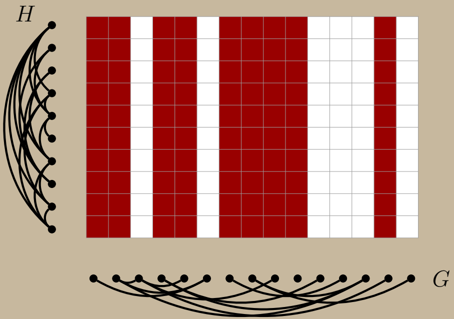
The idea of the proof - Zhu's lemma



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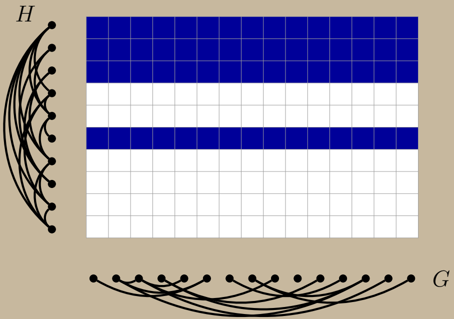

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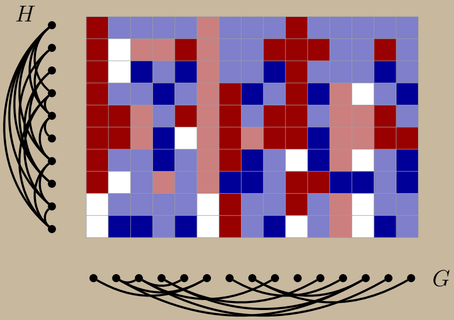

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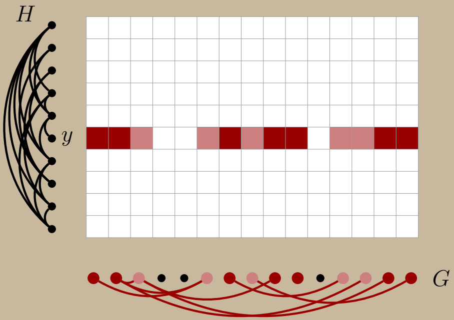

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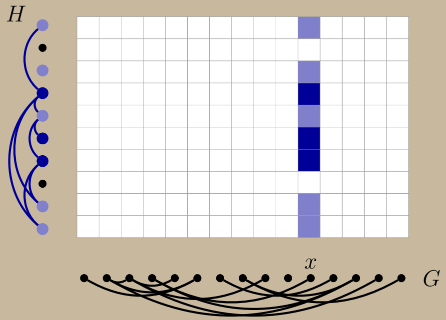

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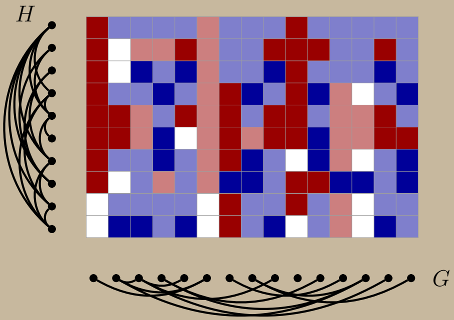
$$|A| + |B| = |U|, \ |A| + |B| + |MA| + |MB| \leq |V(G \times H)|$$



Thank you for your attention!