An Invitation to Nested Recurrence Relations

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Agenda

- ❖ Basics about nested recursions and the properties of their solutions
- ❖ Highlights about some early recursions and their more recent generalizations
- ❖ From interesting **individuals** to **families** with similar behaviour: focus on generalized Conolly recursion
- ❖ Tree-based combinatorial interpretation for solutions to (generalized) Conolly families of recursions
- ❖ Ceiling function solutions to (generalized) Conolly families of recursions

A nested recursion…

Loosely speaking, any recursion where at least one of the arguments contains a term of the recursion.

Some early examples:

R(n) = R(n-R(n-1)), Ics: some finite set (Golomb, ca. 1980?)

R(n) = R(n-R(n-1)) + 1, Ics: R(1) = 1 (Golomb, ca. 1986?)

R(n) = n- R(R(n-1)), Ics: 1 (Hofstadter G, GEB 1979)

R(n) = R(n-R(n-1)) + R(R(n-1)), Ics: 1, 1 (Hofstadter-Conway, 1988)

R(n) = R(n-R(n-1)) + R(n-R(n-2)), Ics: 1,1 (Hofstadter Q, GEB 1979)

R(n) = R(n-R(n-1)) + R(n-1-R(n-2)), Ics: 1,2 (Conolly, 1987)

R(n) = R(n-1-R(n-1)) + R(n-2-R(n-2)), Ics: 1,1,2 (Tanny, 1992)

A solution to a nested recursion is…

Any infinite sequence that satisfies the recursion. No guarantee that a solution exists. What can go wrong?

❖ Try to evaluate the recursion at a **negative** argument:

R(n) = R(n-R(n-1))+1, Ics: 1,4. Then R(3) = R(3-R(2))+1 =R(-1)+1. The sequence terminates ("dies") at n = 3.

R(n) = R(n-R(n-1))+R(n-R(n-4)), Ics: 3,1,4,4. Terminates at n=474,767.

R(n) = R(n-19-R(n-3))+R(n-28-R(n-12)), Ics: 1^{29} Terminates at n = 19,517,558.

Find recursions with increasing "mortality" (Ruskey).

❖ Try to evaluate the recursion for a **future** argument:

R(n) = R(R(n-1))+3, Ics: 1. R(2) = 4 and R(3) = R(4)+3.

More on existence of solutions…

- $R(n) = R(n-R(n-1)) + R(n-R(n-2))$, Ics: 1,1 (Hofstadter Q). Computed to n = 12,148,002,000 (Ruskey).
- A recurrence relation exists, that given a set of Ics, the question of whether the sequence dies for that Ics is **not** decidable. Later this morning **Frank Ruskey** will discuss such an example. (Celaya and Ruskey, 2012)
- Existence (and behaviour) of the solution to a nested recursion can be highly sensitive to the parameters and to the set of Ics.

Solving a nested recursion…

Nesting makes recursions highly resistant to usual techniques for solving difference equations.

Initial focus on solving **individual** recursions; proof technique usually (multi-statement) induction.

Recent work on solving **families** of recursions characterized by one or more parameters using alternate proof techniques.

Closed form solutions sometimes available:

- ❖ R(n) = R(n-R(n-1))+1, Ics: 1: R(n) = fl{[1+fl{√(8n)}]/2}.
- ❖ R(n) = n- R(R(n-1)), Ics: 1: R(n) = fl{(n+1)/α}, α golden mean.
- ❖ R(n) = R(n-R(n-1))+R(n-2-R(n-3)), Ics: 1,1: R(n) = cl{n/2}.
- ❖ R(n) = R(n-R(n-2))+R(n-4-R(n-6)), Ics: 1,2,2,2,3,4: R(n) = cl{n/4}+cl{(n-1)/4}.

Solution properties can vary greatly…

- Preceding closed forms indicate that some solutions are increasing with successive terms differing by 0 or 1 (call these **slow growing** or **slow**). Not surprisingly, the most is known about nested recursions with such solutions.
- More generally, some solutions display well-behaved, discernible structure. Sometimes the solution is **periodic** or **“quasi-periodic”.**
- Some solutions initially appear chaotic, but subsequent analysis uncovers some underlying structure.
- Some solutions are wild, with no hint of any structure, yet appear to remain well defined for all n. How do we demonstrate this?

R(n) = R(n-R(n-1)) (Golomb)

One of the earliest examples of a nested recursion. Need to provide appropriate Ics to ensure a solution.

Every solution is eventually **periodic**, with all its values taken from those in the Ics (Cheng, 1981, PhD student of Golomb). Cheng calls these Golomb sequences.

Ics: 1,3,2 yields R(4) = R(2) = 3; R(5) = R(2) = 3; R(6) = R(3) = 2; R(7) = R(5) = 3; R(8) = R(5) = 3; R(9) = R(6) = 2; sequence is {1,3,2,3,3,2,3,3,2,…} so eventually periodic with period 3 and cycle (3,3,2).

Period of the solution sequence can be larger than the largest value among the Ics. Here is an example: take Ics: 6,9,3,6,3,3,6,9,6,6,3,6. This yields a sequence that is periodic with period 12, with the Ics as the cycle.

$R(n) = R(n-R(n-1))+1$, $R(1) = 1$ (Golomb)

Early recursion, closed form solution: $R(n) = fl\{[1+\sqrt{(8n)}]/2\}$.

Solution: 1,2,2,3,3,3,4,4,4,4,…; each positive n appears n times. Sequence is slow. First proof by induction.

"This furnishes an important example of a recursion which looks as "strange" as several others that we have considered, but where the resulting sequence is completely regular and predictable. It is a challenging unsolved problem to categorize those "strange" recursions which have well-behaved, closed-form solutions." (Golomb, ca. 1986?) – **Still true today!!**

$R(n) = R(n-s-R(n-1))+1$, Ics: $1^{s+1}2^{s+2}3$: Closed form slow solution $R(n) = fl\{(\sqrt{8(n+s(s+1)/2)}+1)/2\}-s$. Each positive n appears n+s times. Special case of more general result which is proved by tree methodology. (Isgur, Kuznetsov, Tanny, 2012)

$R(n) = n - R(n-R(n-1))$, $R(1) = 1$ (Hofstadter G)

Solution is slow, with **Fibonacci** connection: $R(F_{n+1}) = F_n$

Frequency sequence is Fibonacci string: 2122121221… generated by morphism 2→21 and 1→2, starts at 2. More on morphisms by **Marcel Celaya** soon.

n=	1	2	3	4	5	6	7	8	9	10
R(n+0)	1	1	**2**	3	**3**	4	4	**5**	6	6
R(n+10)	7	8	**8**	9	9	10	11	11	12	12
R(n+20)	**13**	14	14	15	16	16	17	17	18	19
R(n+30)	19	20	21	**21**	22	22	23	24	24	25
R(n+40)	25	26	27	27	28	29	29	30	30	31

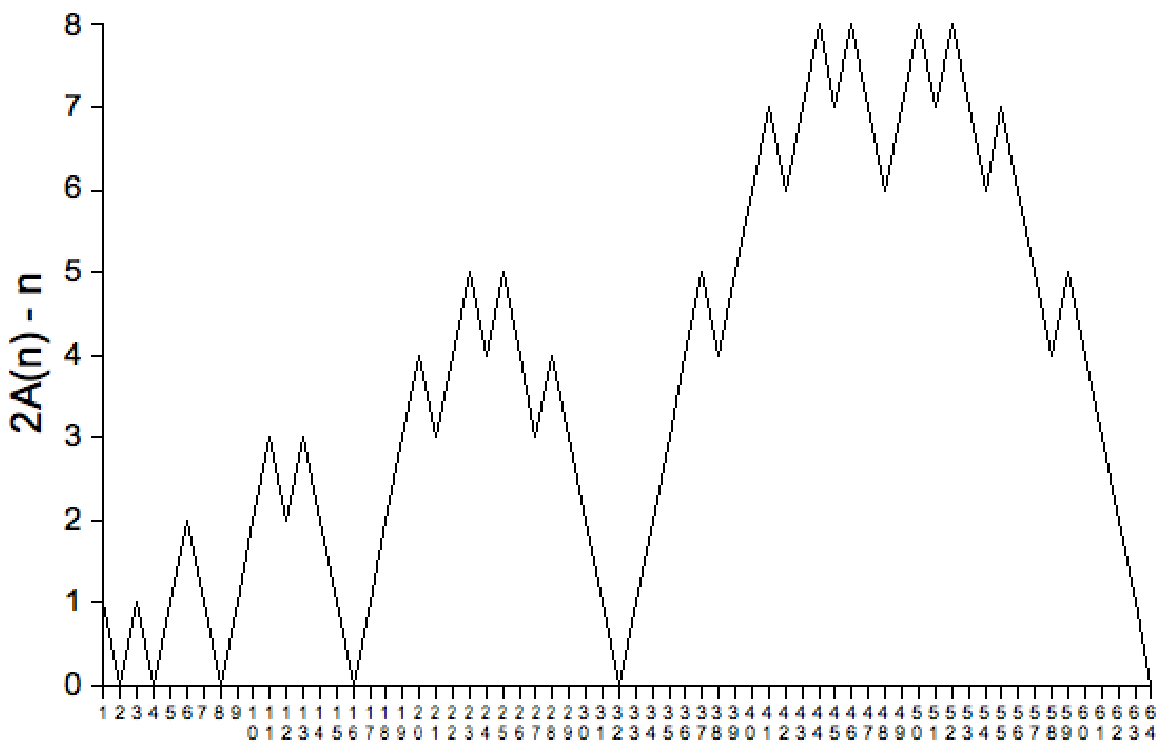
$R(n) = R(n-R(n-1)) + R(R(n-1))$, Ics: 1,1 (Conway-Hofstadter-Newman-$10K)

Early recursion, $R(2^n) = 2^{n-1}$ .Interesting story too!

n=	1	2	3	4	5	6	7	8	9	10
R(n+0)	1	1	2	2	3	4	4	4	5	6
R(n+10)	7	7	8	8	8	8	9	10	11	12
R(n+20)	12	13	14	14	15	15	15	16	16	16
R(n+30)	16	16	17	18	19	20	21	21	22	23
R(n+40)	24	24	25	26	26	27	27	27	28	29
R(n+50)	29	30	30	30	31	31	31	31	32	32
R(n+60)	32	32	32	32	33	34	35	36	37	38
R(n+70)	38	39	40	41	42	42	43	44	45	45
R(n+80)	46	47	47	48	48	48	49	50	51	51
R(n+90)	52	53	53	54	54	54	55	56	56	57

Another view of the Conway-Hofstadter-Newman sequence

Repetition of basic structure within intervals of length 2^n.



Generalizations of $10K sequence

New Ics: $R(n) = R(n-R(n-1))+R(R(n-1))$, Ics: 1^{k+1} (Newman-Kleitman, 1992)

-Solution **slow growing**, with role of powers of 2 played by another class of sequences parameterized by k: $E_n = E_{n-1} + E_{n-k}$ with $E_1 = \ldots = E_k = 1$, then $R(E_n) = E_{n-k}$ for $n > k$. For k=1, $E_n = 2^n$, for k=2 E_n= Fibonacci numbers.

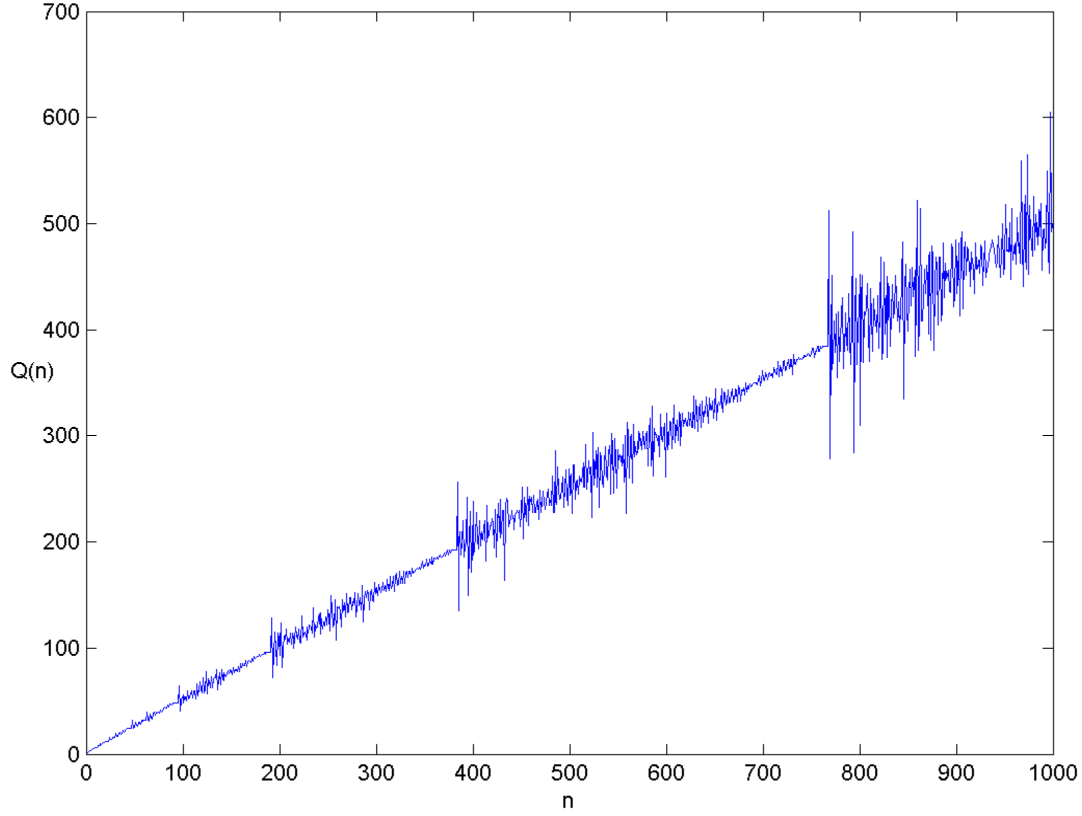
Increase degree of nesting: $R(n) = R(n-R(R(n-1))) + R(R(R(n-1)))$, Ics: 1,1 (Grytczuk, 2004)

-Solution **slow growing**, role of powers of 2 played by Fibonacci sequence E_n but now $R(E_n)= E_{n-1}$. For higher nesting k>3 analogous results with same recursion $E_n = E_{n-1} + E_{n-k}$ with $E_1 = \ldots = E_k = 1$.

$R(n) = R(n-R(n-1))+R(n-R(n-2))$, Ics: 1,1 (Hofstadter Q)

n =	1	2	3	4	5	6	7	8	9	10
Q(n + 0)	1	1	2	3	3	4	5	5	6	6
Q(n +10)	6	8	8	8	10	9	10	11	11	12
Q(n +20)	12	12	12	16	14	14	16	16	16	16
Q(n +30)	20	17	17	20	21	19	20	22	21	22
Q(n +40)	23	23	24	24	24	24	24	32	24	25
Q(n +50)	30	28	26	30	30	28	32	30	32	32
Q(n +60)	32	32	40	33	31	38	35	33	39	40
Q(n +70)	37	38	40	39	40	39	42	40	41	43
Q(n +80)	44	43	43	46	44	45	47	47	46	48
Q(n +90)	48	48	48	48	48	64	41	52	54	56

Alternating chaos and quiet in Q



R(n) = R(n-R(n-1))+R(n-R(n-2)): Alternative Ics make a big difference!

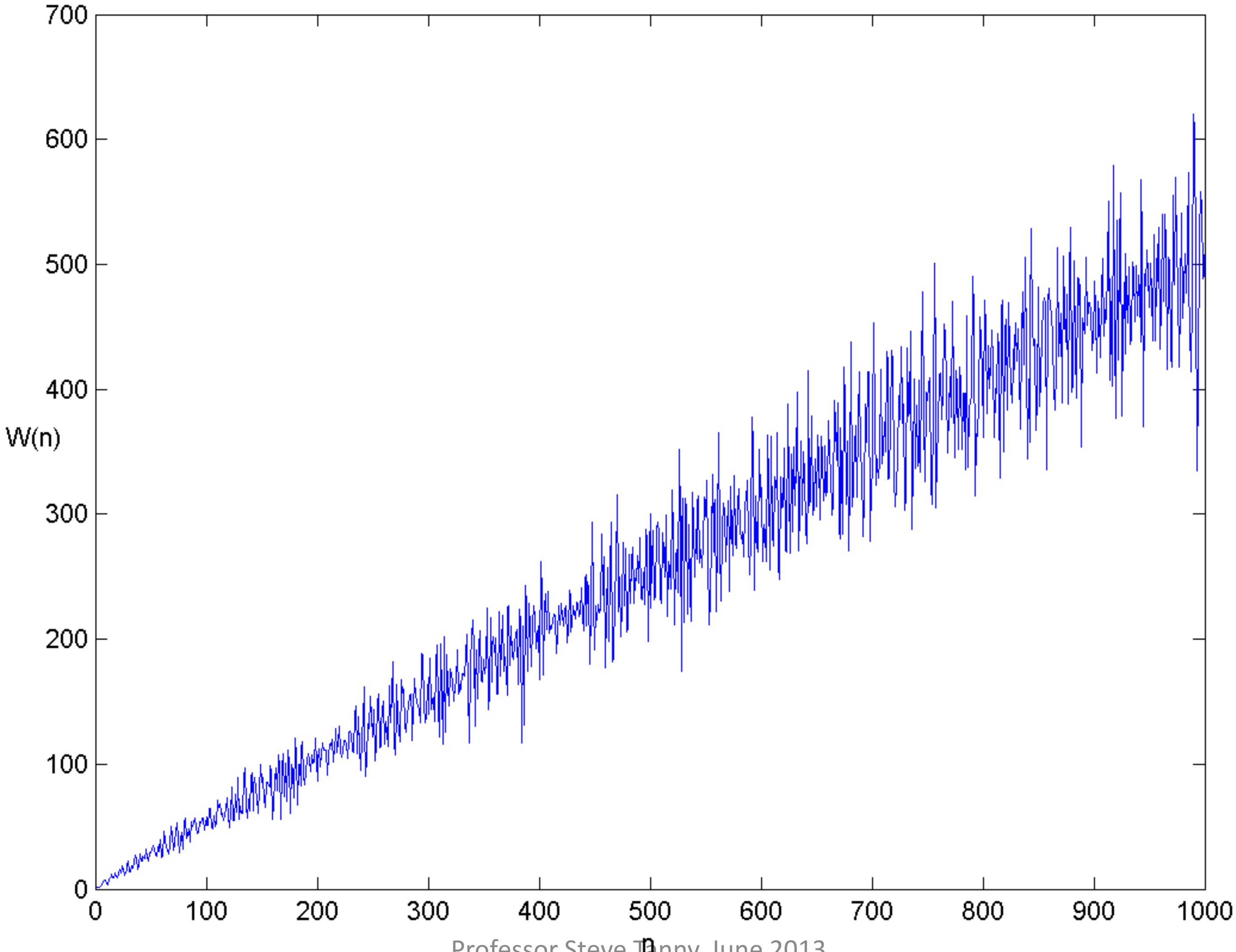
Ics: 3,2,1: For k≥1, solution is: R(3k+1)= 3, R(3k+2)= 3k+2, R(3k) = 3k-2. (Golomb). Example shows sensitivity of nested recursion solutions to Ics. Call behaviour of this solution **"quasi-periodic"** of period 3.

Generate quasi-periodic sequences of **any** period, e.g., Ics: 0,0,2,4,2,4,4,8 give quasi-periodic solution of period 4: R(4k)= 4k, R(4k+1)= 2, R(4k+2)= 4k, R(4k+3)= 4.

Infinite number of Ics (Ruskey, 2011): let R(n) = 0 for n<0, R(0) = R(3) = 3, R(1) = R(4) = 6, R(2) = 5, R(5) = 8. Then R(n) is well defined and for all k≥0, R(3k) = 3, R(3k+1) = 6, and R(3k+2) = F_{k+5}, where F_n is Fibonacci sequence.

In ongoing work we have developed analogous results to Ruskey's for several other nested recursions.

R(n) = R(n-R(n-2))+R(n-R(n-4)), Ics: 1,1,1,1 (Hofstadter W)



R(n) = R(n-R(n-1))+R(n-R(n-4)), Ics: 1,1,1,1 (Hofstadter V)

n =	1	2	3	4	5	6	7	8	9	10
V(n + 0)	1	1	1	1	2	3	4	5	5	6
V(n + 10)	6	7	8	8	9	9	10	11	11	11
V(n + 20)	12	12	13	14	14	15	15	16	17	17
V(n + 30)	17	18	18	19	20	20	21	21	22	22
V(n + 40)	22	23	23	24	25	25	26	26	27	27
V(n + 50)	28	29	29	29	30	30	31	32	32	33
V(n + 60)	33	34	34	34	35	35	36	37	37	38
V(n + 70)	38	39	39	40	41	41	41	42	42	43
V(n + 80)	43	44	44	44	45	45	46	47	47	48
V(n + 90)	48	49	49	50	51	51	51	52	52	53

V is immortalized in verse

Kellie O’Connor Gutman
Recalling a Collaboration with Greg Huber and Doug Hofstadter

And now, my friends, in poetry,
The lowdown on the function V,
Which calls itself recursively.
My verse will mirror it, you’ll see.
…

Mathematical Intelligencer 23 (3) (2001), 50.

Frequency sequence of V: some data

a =	1	2	3	4	5	6	7	8	9	10
F(a + 0)	4	1	1	1	2	2	1	2	2	1
F(a + 10)	3	2	1	2	2	1	3	2	1	2
F(a + 20)	2	3	2	1	2	2	2	1	3	2
F(a + 30)	1	2	2	3	2	1	2	2	2	1
F(a + 40)	3	2	2	3	2	1	2	2	2	1
F(a + 50)	3	2	2	1	2	2	2	3	2	1
F(a + 60)	2	2	2	1	3	2	2	3	2	1
F(a + 70)	2	2	2	1	3	2	2	1	2	2
F(a + 80)	2	3	2	1	3	2	2	3	2	1
F(a + 90)	2	2	2	1	3	2	2	1	2	2

Rules determine frequency sequence for V

F(a-2)	F(a-1)	F(a)	F(a+1)	F(2a)	F(2a+1)
		1		2	2
		3		3	2
	1	2		1	3
	3	2	3	1	3
	3	2	2	1	3
	3	2	1	1	2
1 or 3	2	2	1	2	1
1 or 3	2	2	2 or 3	2	2
2	2	2	1	1	2
2	2	2	3	1	3

Automata and nested recurrences

Jeff Shallit will talk later this morning about the relation between automata and nested recurrences.

In particular he will show that the frequency sequence of V, which is given by the preceding rules, is 2-automatic. (Shallit, 2011)

Recently we identified family of related recursions with “V-like” solutions: recursions with lcs whose solutions are slow and (eventually) obey similar or analogous frequency sequence rules as those for V. We expect there are more automatons lurking!

R(n)= R(n – R(n-1)) + R(n - 1 – R(n-2)), Ics:1,2 (Conolly, 1989)

n=	1	2	3	4	5	6	7	8	9	10
F(n + 0)	1	2	2	3	4	4	4	5	6	6
F(n + 10)	7	8	8	8	8	9	10	10	11	12
F(n + 20)	12	12	13	14	14	15	16	16	16	16
F(n + 30)	16	17	18	18	19	20	20	20	21	22
F(n + 40)	22	23	24	24	24	24	25	26	26	27
F(n + 50)	28	28	28	29	30	30	31	32	32	32
F(n + 60)	32	32	32	33	34	34	35	36	36	36
F(n + 70)	37	38	38	39	40	40	40	40	41	42
F(n + 80)	42	43	44	44	44	45	46	46	47	48
F(n + 90)	48	48	48	48	49	50	50	51	52	52

Frequency sequence of Conolly sequence is "ruler function"

Like V, the Conolly sequence is slow.

The frequency sequence for the Conolly sequence is very different from V: 1, **2**, 1, **3**, 1, 2, 1, **4**, 1, 2, 1, 3, 1, 2, 1, **5**, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, **6**, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, **7**,…

Frequency sequence is "ruler" function r(n): 1 plus the exponent of 2 in the prime factorization of n (the 2-adic valuation of n).

For each n≥0 initial 2^n terms repeat, but final term increases by 1; it follows that in Conolly sequence there are **2** 2s, **3** 4s, **4** 8s, **5** 16s,…

Constructing a general "family" of Conolly sequences

1. $T_1(n) = T_1(n-1-T_1(n-1)) + T_1(n-2-T_1(n-2))$, Ics: 1,1,2. (Tanny 1992)

1, 1, 2, 2, 2, 3**, 4, 4, 4, 4,** 5, 6, 6, 7, **8, 8, 8, 8, 8**, 9, 10, 10, 11, 12, 12, 12, 13, 14, 14, 15, **16, 16, 16, 16, 16, 16**,..

Frequency sequence: **2**, **3**, 1, **4**, 1, 2, 1, **5**, 1, 2, 1, 3, 1, 2, 1, **6**,.. Each power of 2 in $T_1(n)$ occurs 1 more time compared to Conolly sequence.

This generalizes by introducing a new parameter "s". (Jackson, Ruskey, 2006)

2. $T_s(n) = T_s(n-s-T_s(n-1)) + T_s(n-(s+1)-T_s(n-2))$; Ics: 1^{s+1},2.

Frequency sequence: **s+1**, **s+2**, 1, **s+3**, 1, 2, 1, **s+4**, 1, 2, 1, 3, 1, 2, 1, **s+5**, …Each power of 2 in $T_s(n)$ occurs **s** more times compared to Conolly sequence.

More sequences in general Conolly family: k summands

1. $T(n) = T(n-s-T(n-1)) + T(n-s-1-T(n-2)) + \ldots + T(n-s-(k-1)-T(n-k))$, Ics: 1^{s+1},2,…,k. (Ruskey, Degau, 2009; Higham, Tanny, 1993 for s=1).

Slow solution, frequency sequence ruler function based on k. For k=3, s=0: **1**,1,**2**,1,1,2,1,1,**3**,1,1,2,1,1,2,1,1,3,1,1,2,1,1,2, 1,1,**4**,…

But Ics make big difference: using all 1s yields family of solutions with very different properties.

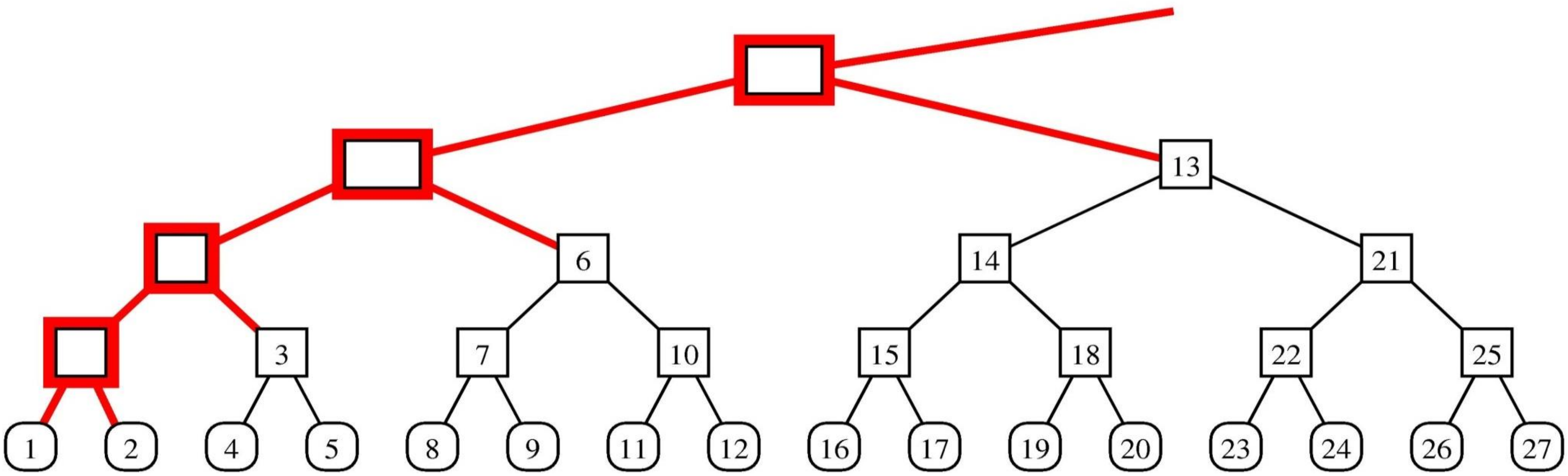
2. $T(n) = T(n-s-T(n-1)) + T(n-s-1-T(n-2)) + \ldots + T(n-s-(k-1)-T(n-k))$, Ics: 1^{s+k}. (Callaghan, Chew, Tanny, 2005).

Solution **not** slow, but partitions into k-1 subsequences where successive terms differ by 0 or k-1.

Leaf counting in infinite labeled binary tree T_0 (Jackson-Ruskey, 2006)

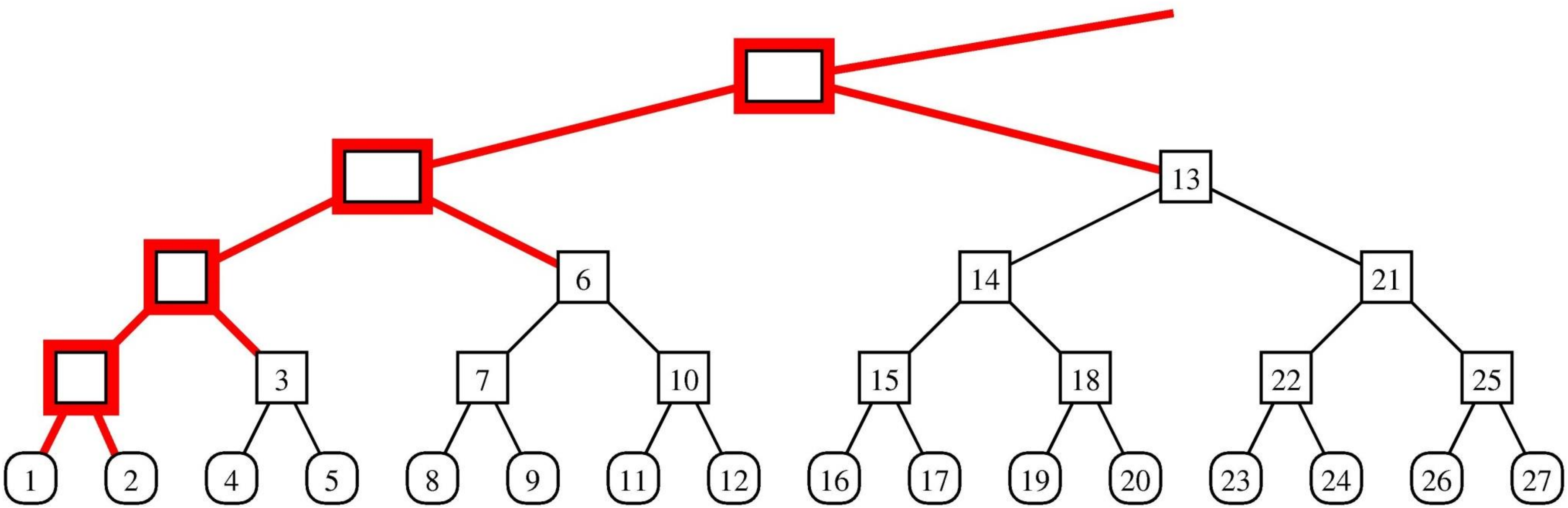
T_0: complete binary trees of sizes 1,1,3,7,…,2^h-1,… labeled in pre-order, joined left to right by an infinite path of "super" nodes.

$L_{T0}(n)$ ≡ no. of nonempty leaves in $T_0(n)$ (T_0 with labels ≤n).



Eureka! Conolly sequence R(n) counts leaves in $T_0(n)$ (tree with labels ≤n)

R(13) = 8 = R(13-R(12)) + R(12-R(11)) = R(13-8) + R(12-7) = R(5)+R(5) = 4+4 =8. 1st (2nd) term counts left (right) leaves. (Jackson-Ruskey, 2006)



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Sketch tree proof method: Leaf counting function solves the Conolly recursion

$L_{T0}(n) \equiv$ number of nonempty leaves in $T_0(n)$ (**leaf counting fn**).

$L_{T0}(n)$ **satisfies Ics** (n=1,2). For **n>2** $L_{T0}(n)$ **satisfies the recursion**:

$L_{T0}(n) = L_{T0}(n- L_{T0}(n-1)) + L_{T0}(n-1- L_{T0}(n-2))$. First (second) summand counts nonempty left (right) leaves. Sketch argument:

$L_{T0}(n- L_{T0}(n-1)) \equiv$ number of nonempty leaves in $T_0(n- L_{T0}(n-1))$.

"Prune" $T_0(n)$ by removing last row, create binary tree $PT_0(n)$, show $PT_0(n)= T_0(n- L_{T0}(n-1))$; pruning operation corresponds to subtraction of $L_{T0}(n-1)$ from argument n. **N.B**.: Infinite binary tree with bottom level removed is infinite binary tree.

$L_{PT0}(n) \equiv$ number of nonempty leaves of $PT_0(n) = L_{T0}(n- L_{T0}(n-1))$

Key observation: a penultimate node of $T_0(n)$ ends pruning as a nonempty leaf of $PT_0(n)$ if and only if its left child in $T_0(n)$ was nonempty.

Tree-based solutions for generalized Conolly family of nested recursions

Add parameters to recursion. Modify tree structure or labelling. Solutions count leaves or cells in leaves.

1. $R_s(n) = R_s(n-s-R_s(n-1))+R_s(n-(s+1)-R_s(n-2))$; Ics: $1^{(s+1)}2$ (Jackson-Ruskey, 2006). $s \geq 0$.
2. $R(n) = R(n-s-R(n-j))+R(n-s-j-R(n-2j))$; Ics: from related tree (Isgur, Reiss, Tanny, 2009). $j \geq 1$.
3. $R(n) = R(n-s-R(n-j))+R(n-s-2j-R(n-3j))$; Ics: from related tree (Isgur, Reiss, Tanny, 2009). $j \geq 1$.
4. $R(n) = R(n-s-R(n-j))+R(n-s-j-m-R(n-2j-m))$; Ics: from related tree (Isgur, 2012). $0 \leq m \leq j$. Generalizes 2 and 3.
5. $R(n) = R(n-s-R(n-j))+R(n-s-j-R(n-2j+q))$; Ics: from related tree (Isgur, 2012). Generalizes 2 and 3. $0 \leq q \leq j$.

And many more, including recursions with k summands.

Nested recursions, simultaneous parameters and tree superpositions

Mustazee Rahman, whose talk will follow this one, will provide many more details about the nature of the tree-based methodology for solving the preceding and other nested recursions.

Asymptotic behaviour of solutions for generalized Conolly nested recursions

Parameters positive or non-negative integers; k is “arity”; **p** = (p_1, p_2, …,p_k) is “order” (all p_i = p, recursion has order p); if **ν** = 0 then homogeneous. Assume **c** lcs.

If A(n) is any solution (not necessarily slow) such that A(n)/n tends to a limit L>0, then L= (k-1)/$\sum p_i$. If p_i = p for all i, then L=(k-1)/kp. (If k = 2, p = 1, then L = ½; Hofstadter sequence Q(n) seems to have this asymptotic limit.)

$$R(n) = \sum_{i=0}^{k} R\left(n - s_i - \sum_{j=0}^{p_i} R(n - a_{ij}) \right) + \nu$$

Ceiling functions and their sums solve certain generalized Conolly recursions

R(n) = R(n-s-R(n-a))+R(n-t-R(n-b)), a, b both odd and 2(s+t) = a+b. Solution is cl[n/2] (appropriate Ics). And conversely! (Erickson, Isgur, Jackson, Ruskey, Tanny, 2012)

More generally: Define the sum of ceiling function sum **C(n) = ∑cl[(n-i)/2j]**, sum on i=0,1,…,j-1. Then C(n) satisfies the nested recursion **R(n) = R(n-s-R(n-a))+R(n-t-R(n-b))** with appropriate Ics if and only if the following conditions hold: (i) s,t ≡ 0 mod j (ii) a, b ≡ j mod 2j (iii) 2(s+t) = a+b. (Drabek, Isgur, Kuznetsov, Tanny, 2011).

For every q>1, cl[n/q] solves generalized Conolly recursion (appropriate Ics) (Isgur, Kuznetsov, Tanny, 2011). Similar result for **C(n) = ∑cl[(n-i+1)/kj]**, sum on i=1,…,j. (Isgur, 2012)

Now this is not the end…

It is not even the beginning of the end. But it is the end of the beginning. (Winston Churchill)

There is much more to come. **Mustazee Rahman** will take up the story.