Weighted Well-Covered Graphs without Cycles of Lengths 4, 5 and 6

David Tankus, Ariel University, ISRAEL

Joint work with

Vadim E. Levit, Ariel University, ISRAEL

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Maximum Independent Set

- **:** Finding a maximum cardinality independent set is NPC, even if the input is restricted to $K_{1,4}$-free graphs.

- **Minty [1980]:** There exists a polynomial time algorithm which receives a weighted $K_{1,3}$-free graph as its input, and finds a maximum weight independent set in the graph.

w-Well-Covered Graphs

Let $G = (V,E)$ be a graph and $w:V \rightarrow R$ be a weight function. Than G is w-well-covered if all maximal independent sets are of the same weight.

Caro, Ellingham, Ramey [1998]:

The set of all weight functions $w:V \rightarrow R$ for which G is w-well-covered is a vector space.

That vector space is denoted $WCW(G)$.

$w \in WCW(G) \Leftrightarrow G$ is w-well-covered

Well-Covered Graphs

Caro, Sebo, Tarsi [1996]:

Recognizing well-covered $K_{1,4}$-free graphs is co-NPC.

Tankus, Tarsi [1996]:

There exists a polynomial time algorithm which receives a $K_{1,3}$-free graph as its input, and finds WCW(G).

Well-Covered Graphs

Caro, Ellingham, Ramey [1998]:

Recognizing well-covered graphs with a bounded maximal degree can be done in polynomial time.

Finbow, Hartnel, Nowakowski [1993]:

Recognizing well-covered graphs with girth at least 5 can be done in polynomial time.

Finbow, Hartnel, Nowakowski [1994]:

Recognizing well-covered graphs without C_4 and C_5 can be done in polynomial time.

Well-Covered Graphs

Levit, Tankus[2011]:

The following problem is polynomial:

Input: A graph G without C_4, C_5, C_6 and C_7.

Output: WCW(G).

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Main Result

Levit, Tankus[2012]:

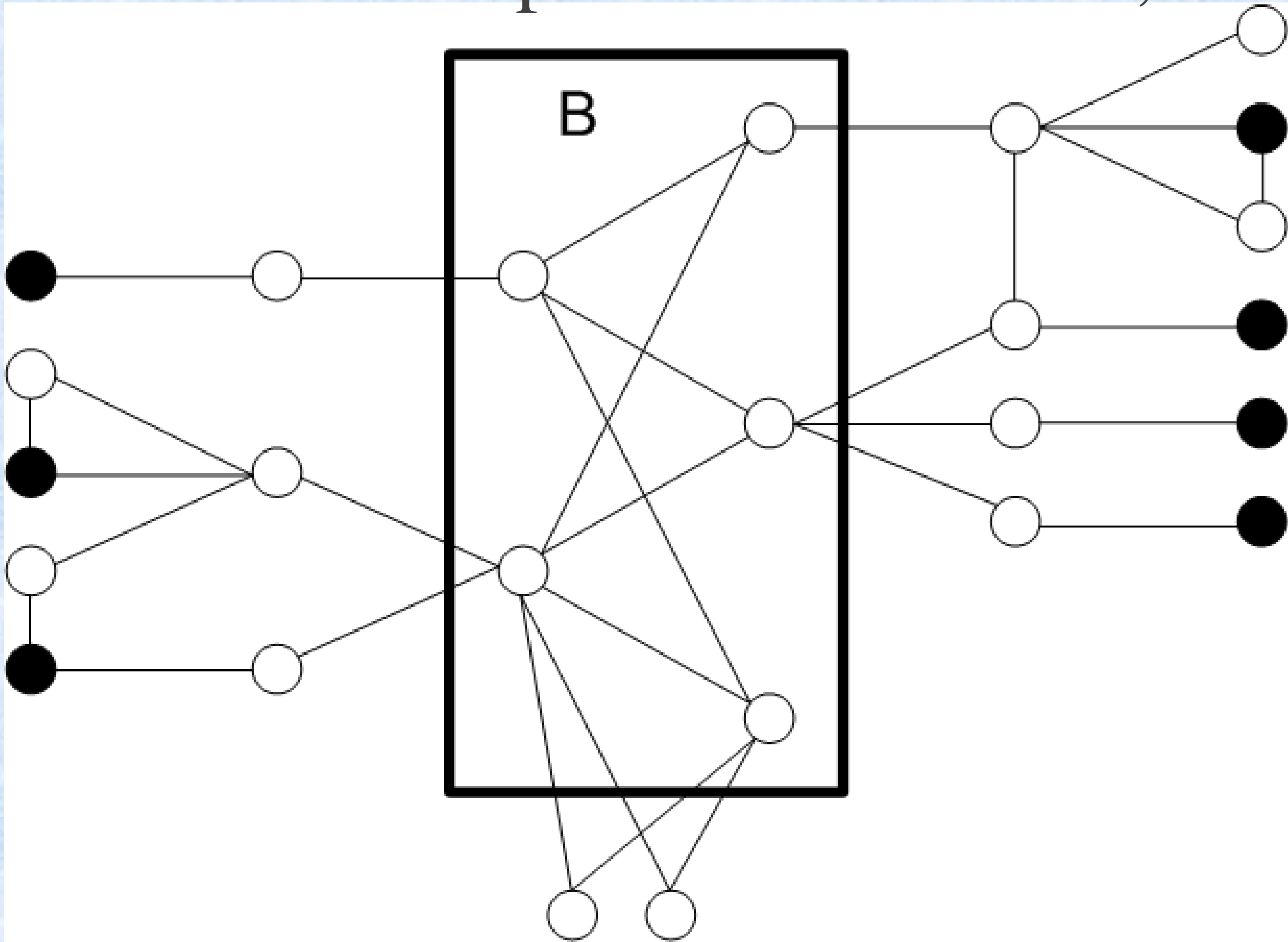
The following problem is polynomial:

Input: A graph G without C_4, C_5 and C_6.

Output: WCW(G).

Generating Subgraph

Let B be an induced complete bipartite subgraph of G with vertex sets of bipartition B_X and B_Y. Then B is generating if there exists an independent set S of G, such that $S \cup B_X$ and $S \cup B_Y$ are both maximal independent sets in the graph.

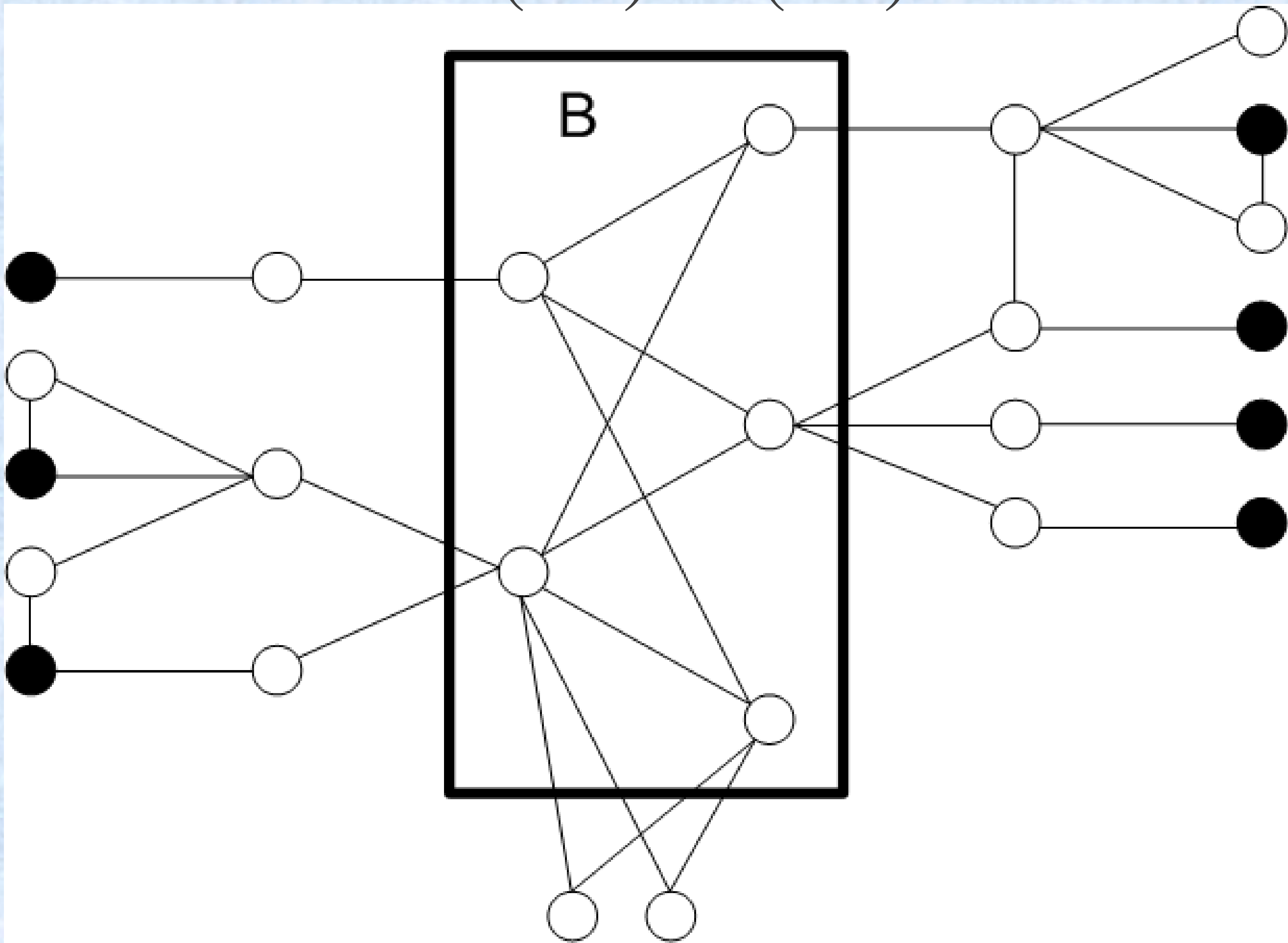


Generating Subgraph

B produces the constraint $w(B_X)=w(B_Y)$.

For every weight function $w:V\rightarrow R$, if $w\in WCW(G)$ then w satisfies the constraint $w(B_X)=w(B_Y)$.

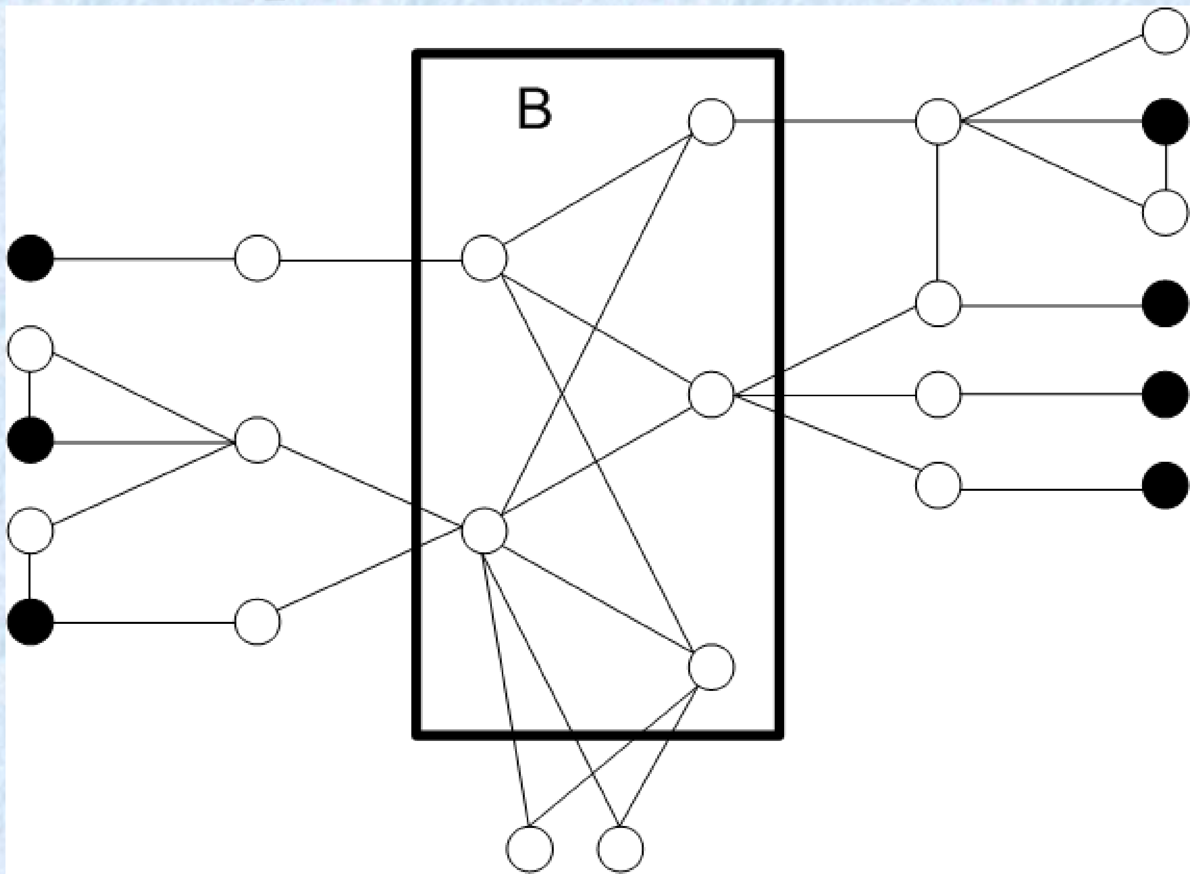
The set S is a witness of B.



Conclusion

Let B be an induced complete bipartite subgraph of G with vertex sets of bipartition B_X and B_Y. The following conditions are equivalent:

- B is generating.
- There exists an independent set $S \subseteq N_2(B)$ which dominates $N(B_X) \Delta N(B_Y)$.



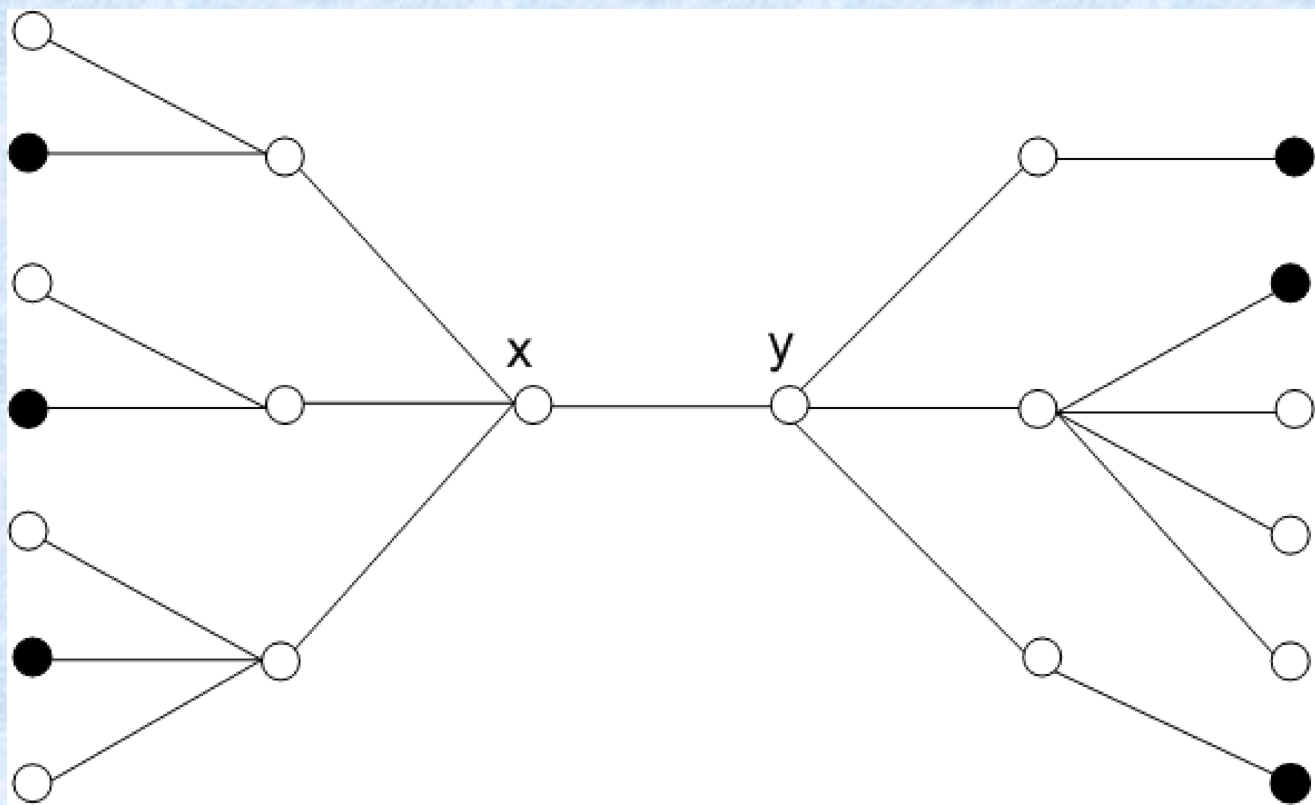
Relating Edge

Brown, Nowakowski, Zverovich[2007]:

The following problem is NPC:

Input: A graph $G=(V,E)$, and an edge $xy \in E$.

Question: Is xy a relating edge?



Relating Edge

Levit, Tankus [2009]:

The following problem is polynomially solvable:

Input: A graph $G=(V,E)$ which does not contain cycles of length 4 or 6, and an edge $xy \in E$.

Question: Is xy a relating edge?

Generating Subgraph

Levit, Tankus [2009]:

The following problem can be solved in polynomial time:

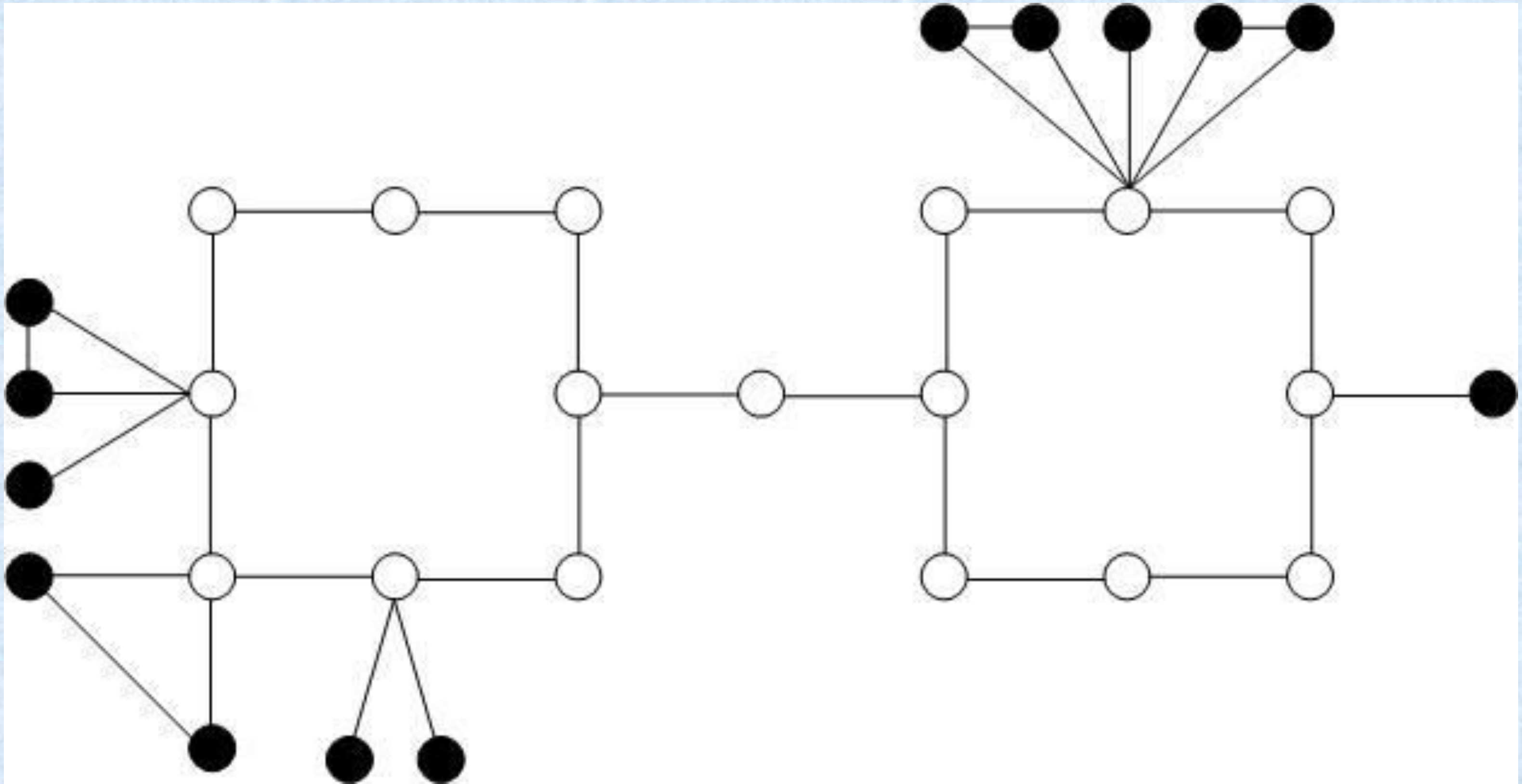
Input: A graph G=(V,E) with no cycles of length 4, 6 or 7 and an induced complete bipartite subgraph B of G.

Question: Is B generating?

Notations and Definitions

L(G) The set of all vertices v in G such that one of the following holds:

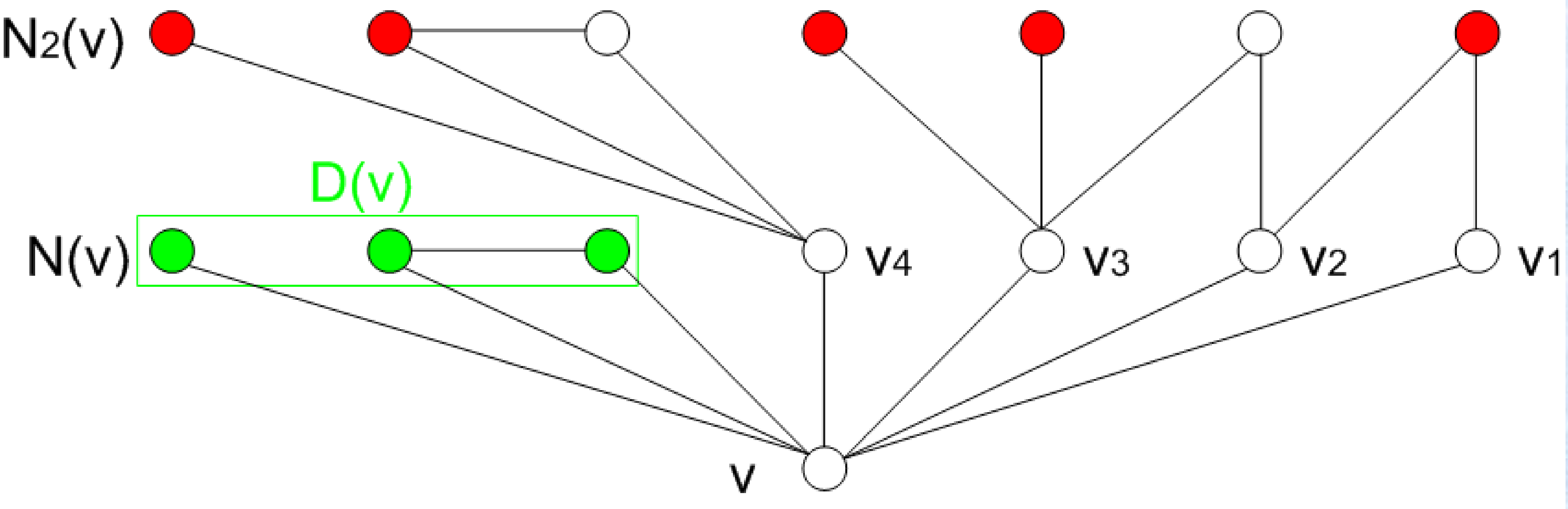
- d(v)=1
- d(v)=2 and v is on a triangle.



Notations and Definitions

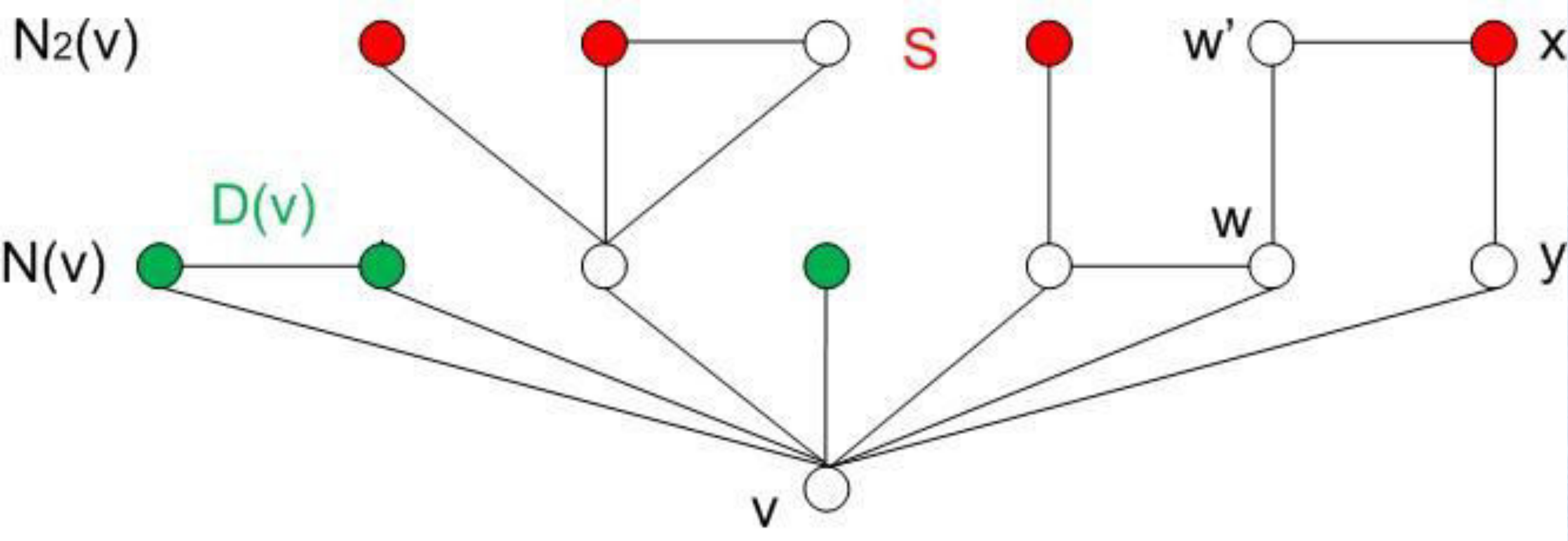
$D(v) = N(v) \setminus N(N_2(v))$

$M(v)$ is a maximal independent set of $D(v)$.



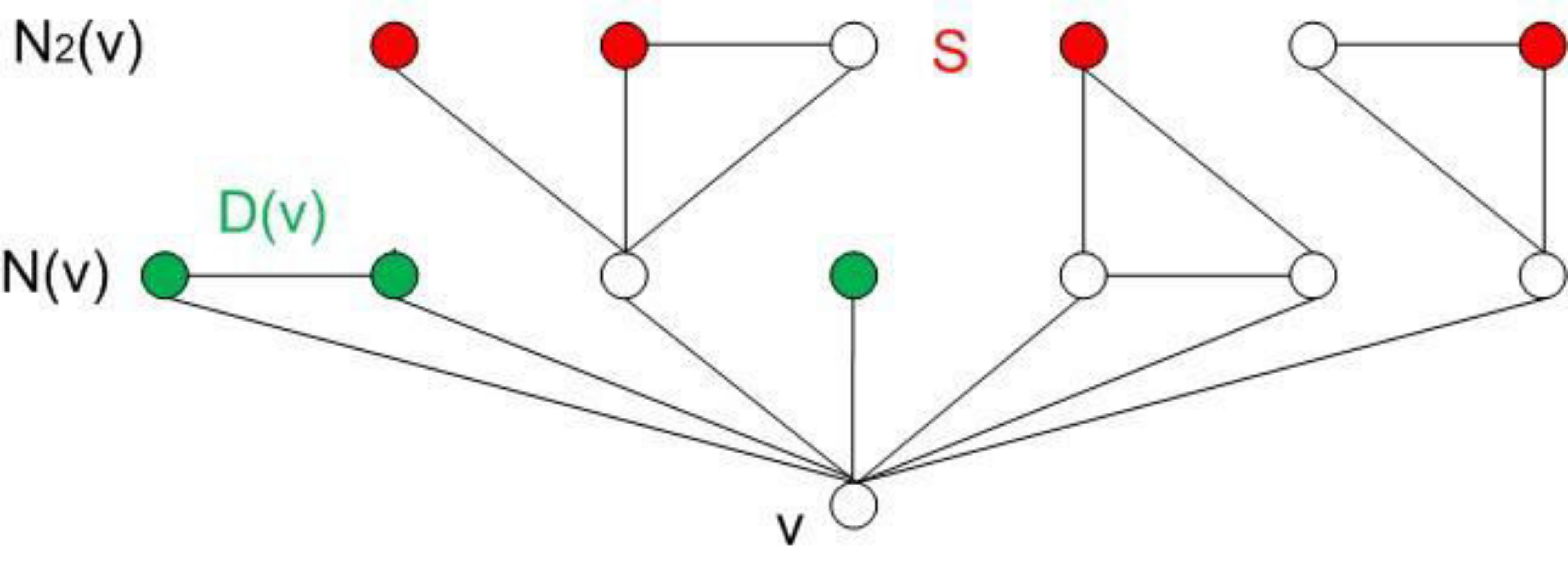
Theorem 1

Let $G = (V,E)$ be a graph without C_5, and let $v \in V$. Then every maximal independent set of $N_2(v)$ dominates $N(v) \cap N(N_2(v)) = N(v) - D(v)$.



Theorem 2

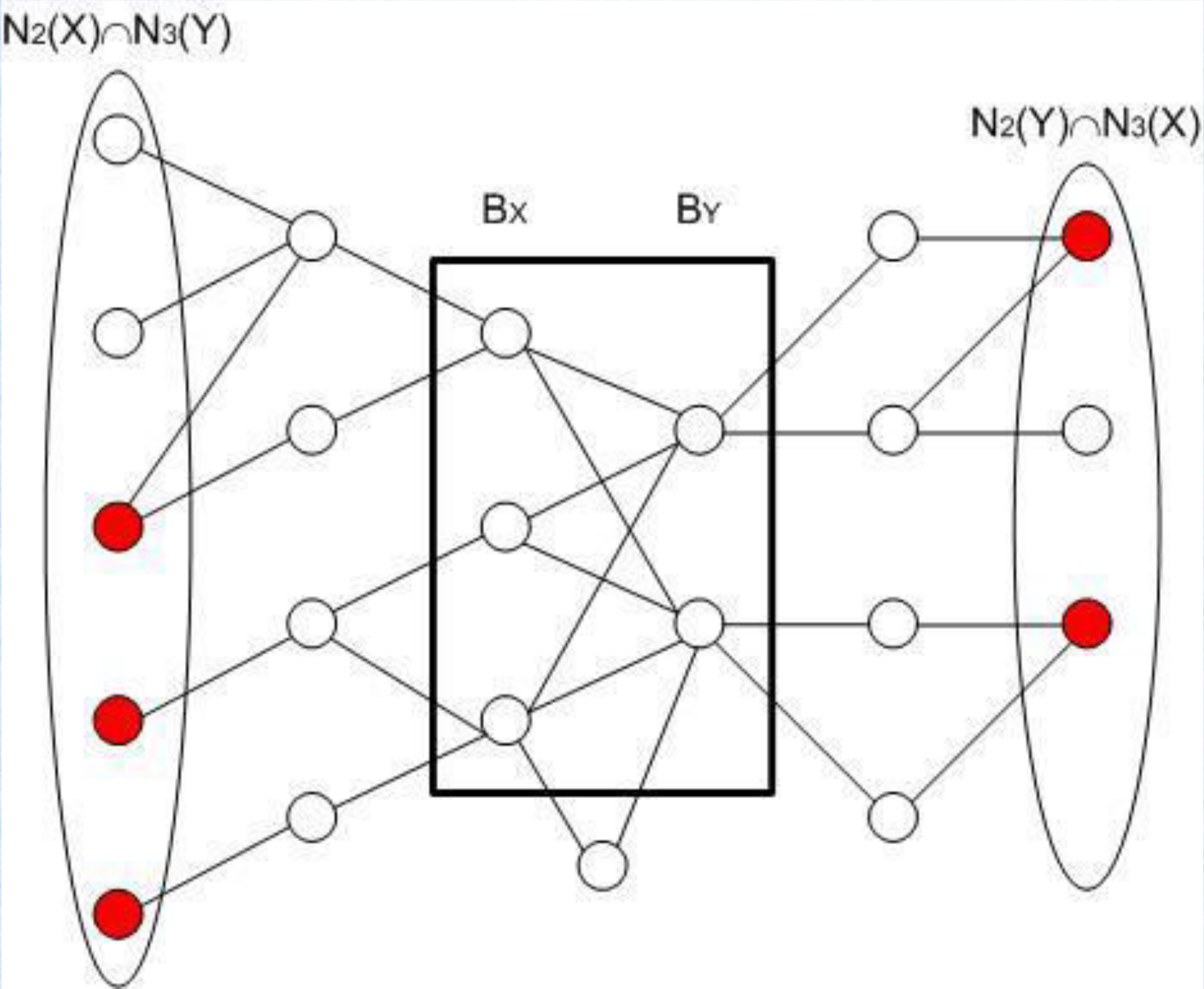
Let $G = (V,E)$ be a w-well-covered graph without C_5. Let $v \in V \backslash L(G)$ s.t. $D(v) \neq \Phi$. Then $w(v)=w(M(v))$.



Theorem 3

Let G be a graph without C_5, C_6 and C_7.
Let B be an induced complete bipartite sub-graph of G.
Then B is generating if and only if $N_2(B)$ dominates $N(B_X) \Delta N(B_Y)$.

Proof



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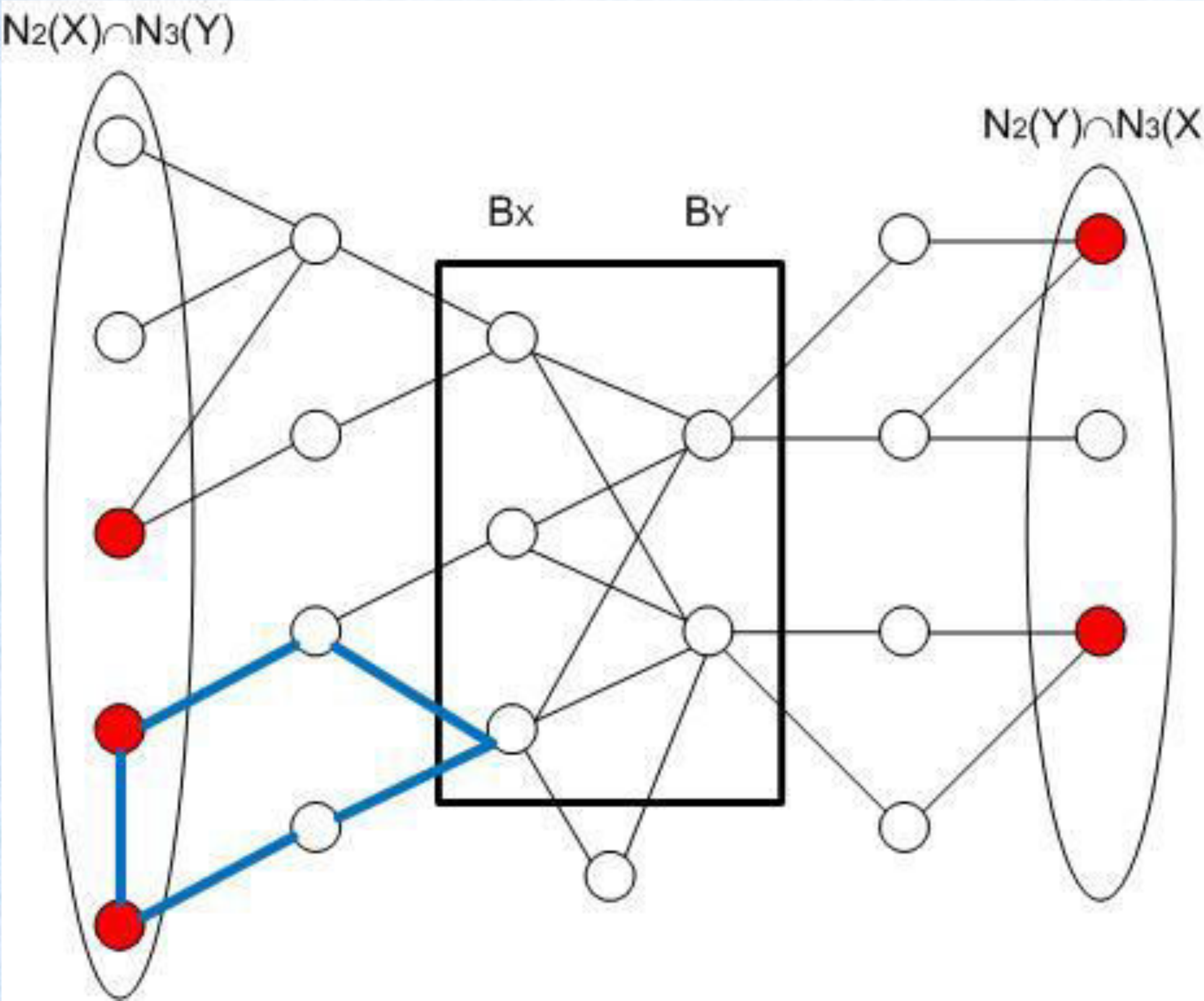

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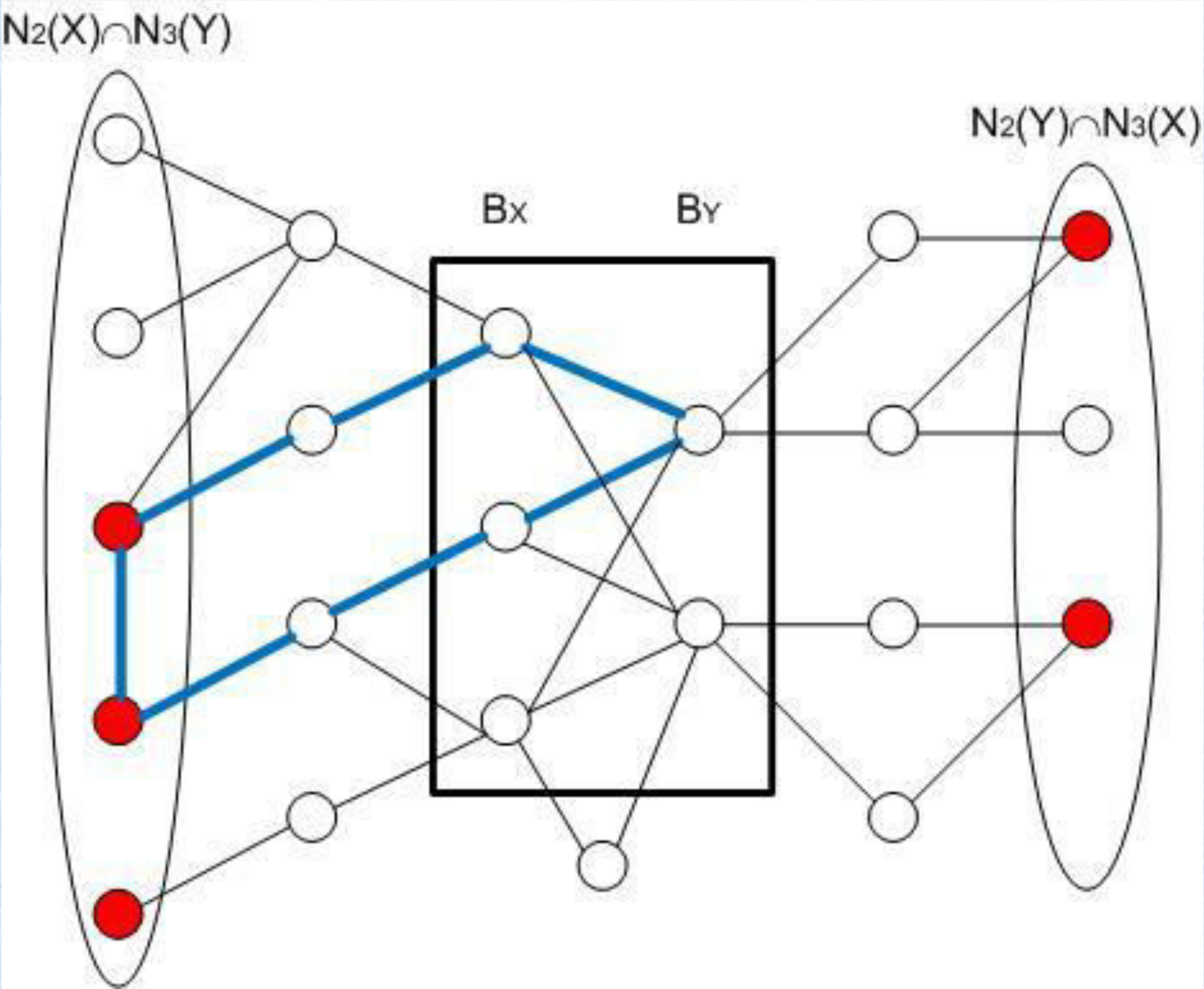

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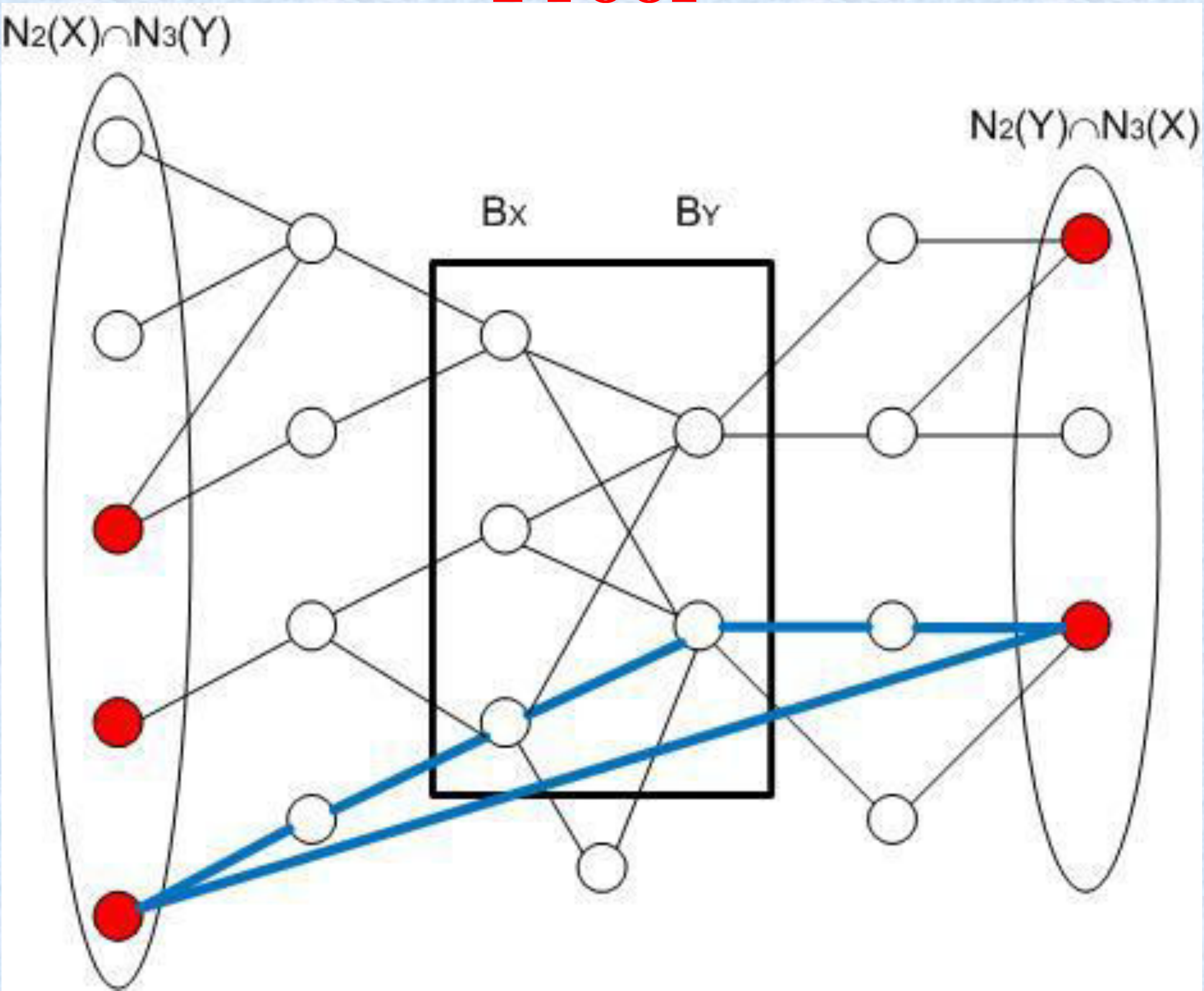
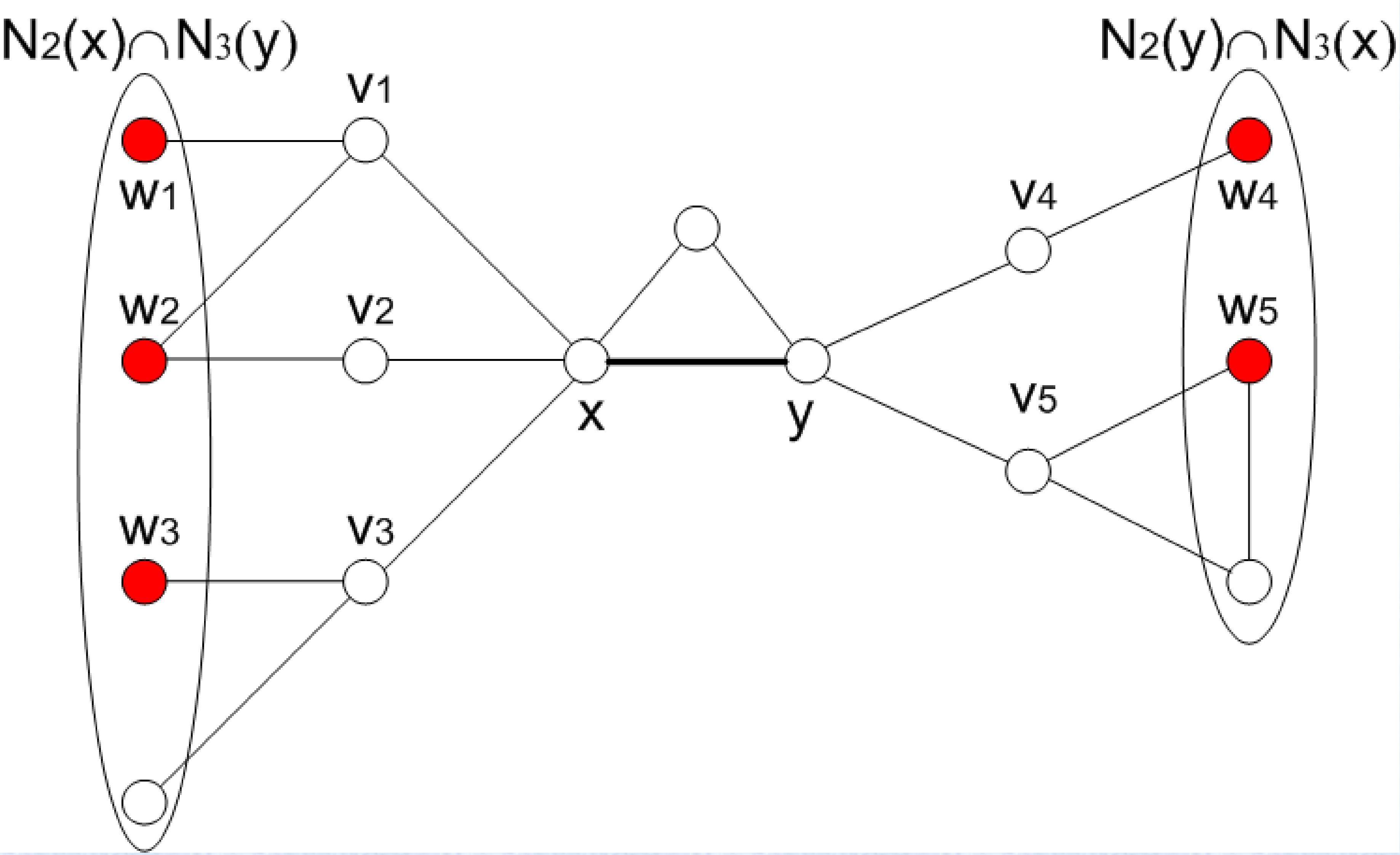


Theorem 4

Let $G=(V,E)$ be a graph without C_5 and C_6 and let $xy \in E$.

Then xy is relating if and only if $N_2(\{x,y\})$ dominates $N(x)\Delta N(y)$.

Proof



Conclusion

Let G=(V,E) be a graph without C_5 and C_6.

If L(G)=Φ then all edges of G are relating.

Theorem 5

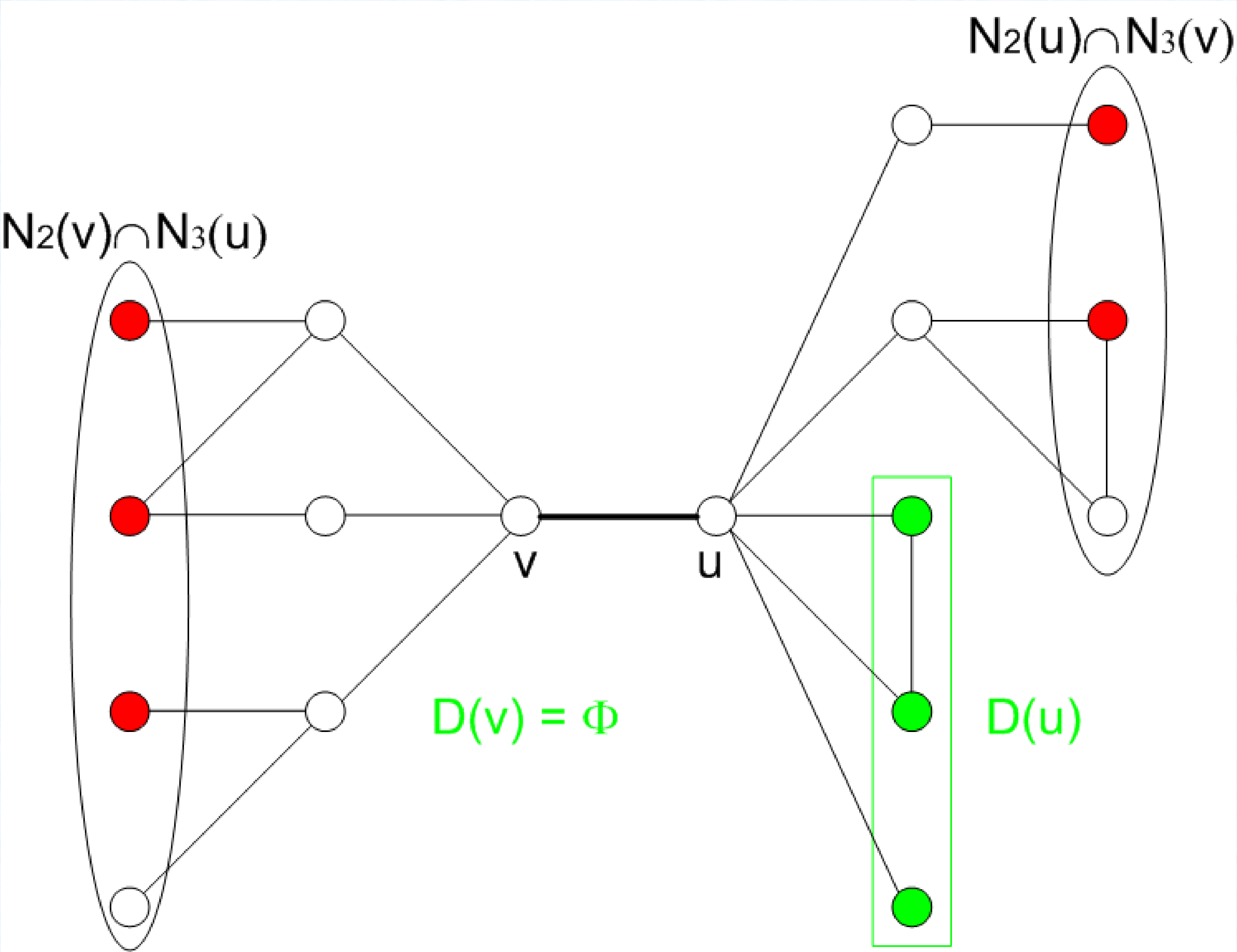
Let G be a w-well-covered graph without C_4, C_5 and C_6.

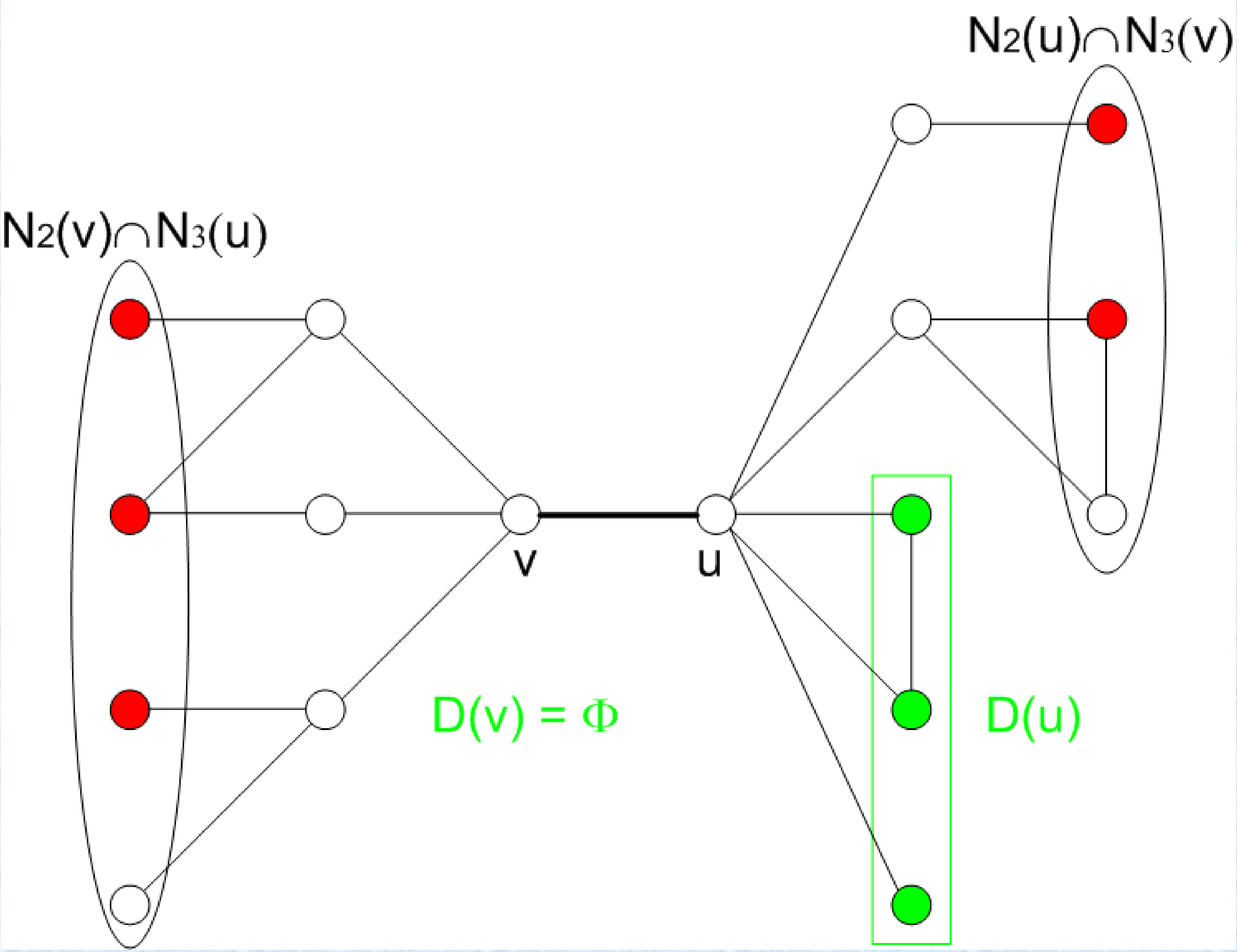
Let $v \in V \setminus L(G)$.

Assume there exists a vertex $u \in \{v\} \cup N(v)$ such that $D(u) \neq \Phi$.

Then $w(v)=w(M(v))$.

Proof





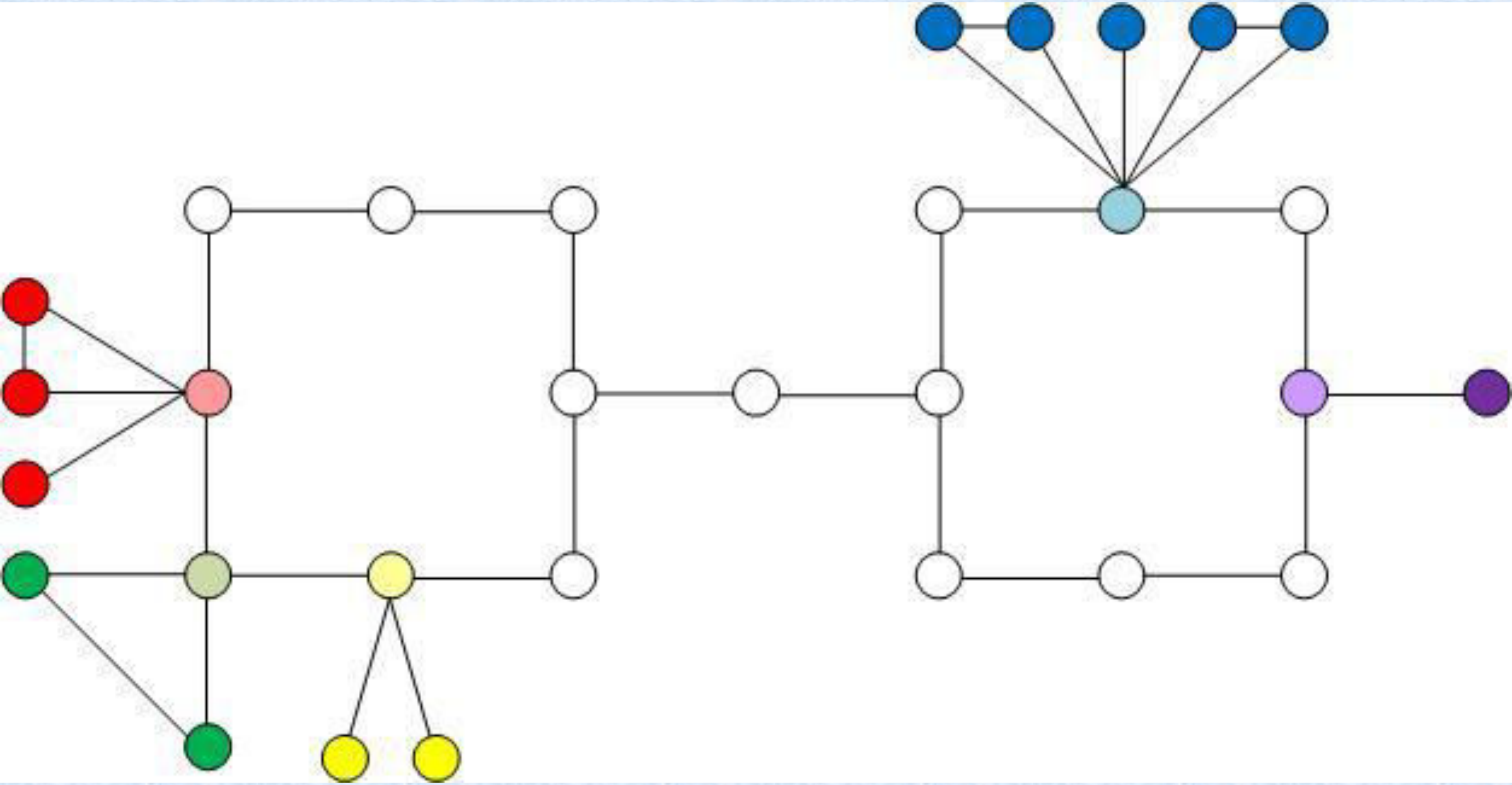
$$\big(\ w(u)=w(M(u))\ ,\ w(u)=w(v)+w(M(u))\ \big) \Rightarrow w(v)=0$$

Theorem 6

Let G be a graph without C_4, C_5 and C_6 such that $L(G) \neq \Phi$.

Then G is w-well-covered if and only if $w(v)=w(M(v))$ for every $v \in V \backslash L(G)$.

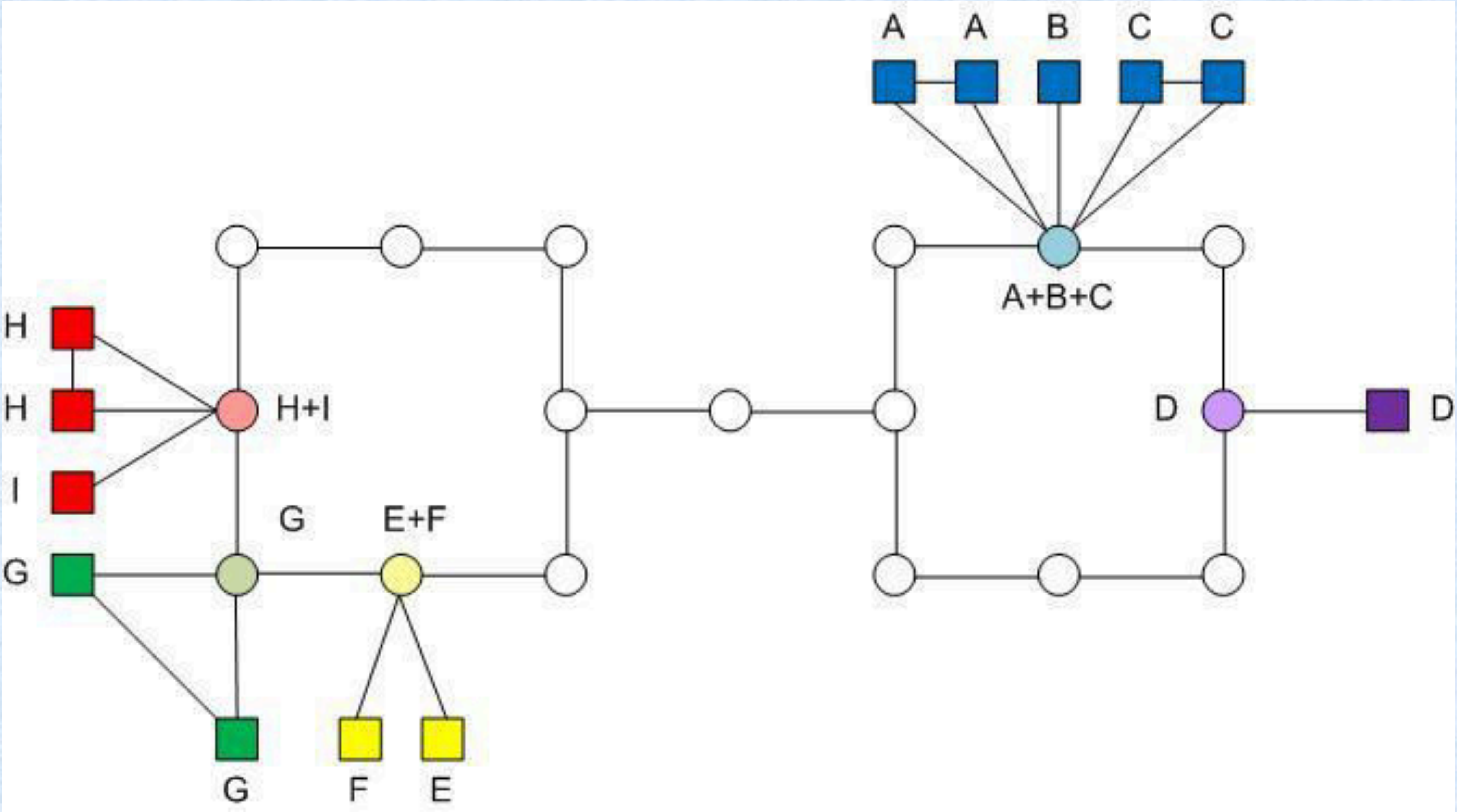
Proof



The weight of every maximal independent set is:

$$\sum_{v \in V \setminus L(G)} w(M(v))$$

Proof



The weight of every maximal independent set is:

$$\sum_{v \in V \setminus L(G)} w(M(v)) = A + B + C + D + E + F + G + H + I$$

Proof

If $\exists u \in \{v\} \cup N(v)$ s.t. $D(u) \neq \Phi$ then the Theorem holds by Theorem 5.

H = The subgraph of G induced by:

$\{x \in V \mid D(x) = \Phi\}$

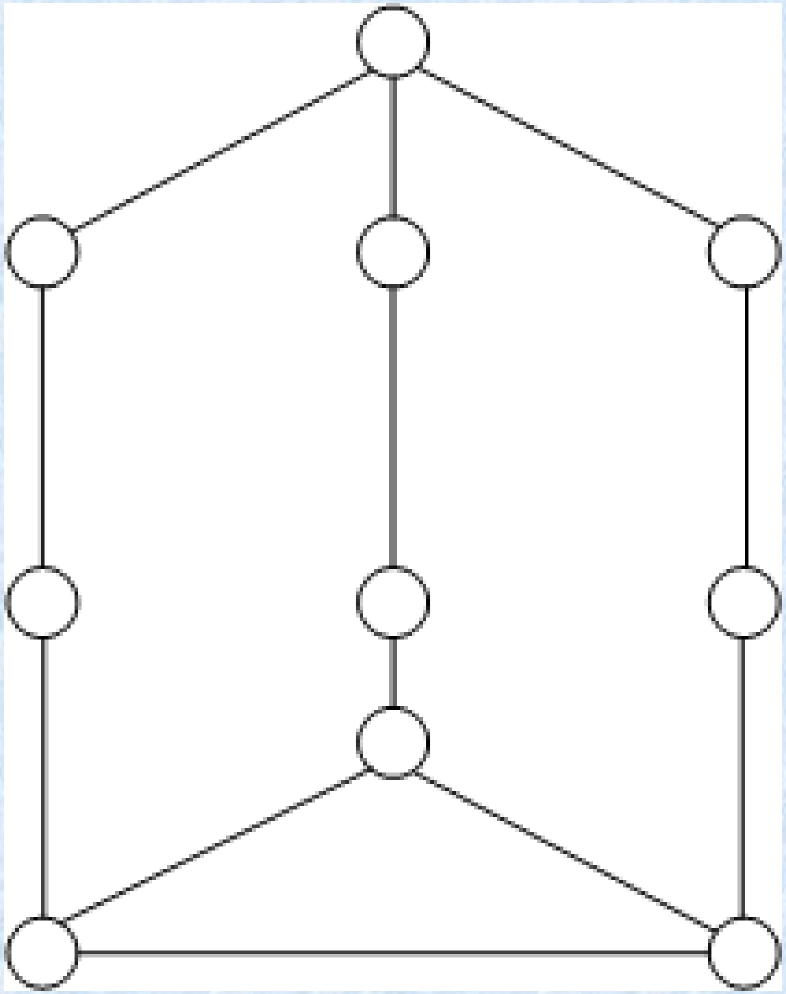
C = The connected component of H which contains v.

Theorem 7

Let G be a graph without C_4, C_5 and C_6, s.t. $L(G)=\Phi$.

If $G=C_7$ or $G=T_{10}$ then G is w-well-covered if and only if $w\equiv k$.

If $G\neq C_7$ and $G\neq T_{10}$ then G is w-well-covered if and only if $w\equiv 0$.



The graph T_{10}

Theorem

Finbow, Hartnel, Nowakowski [1994]:

Let $H = (V,E)$ be a graph without C_4 and C_5. Then H is well-covered if and only if one of the following conditions holds.

- There exists a set $\{v_1,\ldots,v_k\} \subseteq V$ of simplicial vertices such that $|N[v_i]| \leq 3$ for every $1 \leq i \leq k$, and $\{N[v_i] \mid 1 \leq i \leq k\}$ is a partition of V.
- $H=C_7$ or $H=T_{10}$.

Algorithm for Finding WCW(G)

If G is isomorphic to C_7 or to T_{10}

Assign an arbitrary value for k, and denote $w \equiv k$.

Else

Find $L(G)$.

Find a maximal independent set S of $L(G)$.

Assign arbitrary weights to the elements of S.

For each vertex $l \in L(G) \setminus S$

Assign $w(l)=w(N(l) \cap S)$.

For each $v \in V \setminus L(G)$

Find $D(v)$.

Construct a maximal independent set $M(v)$ of $D(v)$.

Assign $w(v)=w(M(v))$

Algorithm for Finding WCW(G)

If G is isomorphic to C_7 or to T_{10}

    Assign an arbitrary value for k, and denote $w \equiv k$.

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    Assign arbitrary weights to the elements of S.

    For each vertex $l \in L(G) \setminus S$

        Assign $w(l) = w(N(l) \cap S)$.

    For each $v \in V \setminus L(G)$

        Find $D(v)$.

        Construct a maximal independent set $M(v)$ of $D(v)$.

        Assign $w(v) = w(M(v))$

$\dim(WCW(G)) = |S|$

Bipartite Graphs without C_4 and C_6

$L(B) = \{v \in V \mid d(v)=1\}$ is independent.

For each $v \in V \setminus L(B)$

$D(v)$ is independent and $D(v)=M(v)$

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Algorithm for Finding WCW(B)

$\dim(WCW(B))=|L(B)|$

Find $L(B)$.

Assign arbitrary weights to the elements of $L(B)$.

For each $v \in V \setminus L(B)$

  Find $D(v)$

  Assign $w(v)=w(D(v))$

Open Question

Levit, Tankus[2012]:

The following problem is polynomial:

Input: A graph G without cycles of length 4, 5 and 6.

Output: WCW(G).

Finbow, Hartnel, Nowakowski [1994]:

Recognizing well-covered graphs without C_4 and C_5 can be done in polynomial time.

Open Question:

Input: A graph G without C_4 and C_5.

Output: WCW(G).

THANK YOU !