Alon
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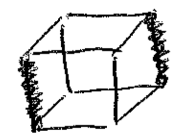
Intro:

Def: Matching M in graph $\equiv$ collection of disjoint edges in graph

$M \equiv$ induced if no edge intersects two edges of M

Ex:

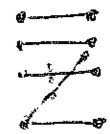


Q (R-S 70's) Let graph G has N vertices and

$G = M_1 \cup \cdots \cup M_r$, M_i induced matching of size t

How large can be r

Intuition: Each matching adds t edges but forbids $O(t^2)$ edges



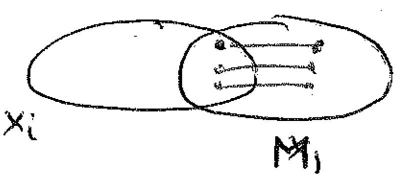
blue edges can not be in G

Hence ratio $\frac{t}{t^2} = \frac{1}{t}$ is small for large t

Observation: $G = M_1 \cup \cdots \cup M_r$, M_i induced of size t

Let $X_i \equiv$ set of vertices covered by edges in M_i, $|X_i| = 2t$

Note $|X_i \cap X_j| \le t$



Cor: (i) $t > N/3$. Then

All $|X_i| > \frac{2N}{3} \Rightarrow |X_i \cap X_j| \ge 4t - N > t$. So $r = 1$

(ii) if $t = (\frac{1}{4} + \varepsilon)N \Longrightarrow r \le C(\varepsilon)$ constant.

Q: 1. If size of M_i is $t = cN$, $G = M_1 \cup \cdots \cup M_r$, M_i induced of size t, $c \equiv$ small const.

show that r is small (should be easy)

2. Can such G have αN^2 edges $\alpha \equiv$ const if $t \longrightarrow \infty$ (by above intuition probably not!)

Additional Motivation

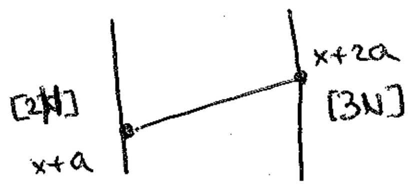
- Additive number th
- computational complexity
- information theory

Additive number theory

Q (E–T '30): $A \subseteq [N]$ with no k-term AP (arithmetic progression). What is $\max |A|$?

Th (Roth $k=3$ 50's; Szemeredi general k 70's): $\forall \delta > 0$ if $A \subseteq [N]$, $|A| = \delta N \Rightarrow A$ contain k-term AP

Construction: $A \subseteq [N]$ with no 3-AP



$\forall x \in [N]$ matching M_x: $x+a$ —— $x+2a$ $\forall a \in A$

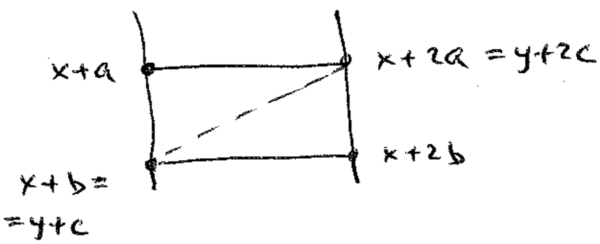
\# of matchings: $r = N$

\# of vertices: $5N$

Size of M_x: $t = |A|$

Claim: M_x induced

$$\begin{cases} x+b = y+c \\ x+2a = y+2c \end{cases} \Rightarrow 2a = b+c \quad \text{3-AP}$$



Says: (i) Upper bounds on ~~induced matching~~ problem $\Rightarrow$ bounds on size of $A \subseteq [N]$ with no 3 AP

(ii) construction of A with no 3 AP $\Rightarrow$ construct. of dense graph which is union of induced M_i and $r = cN$

Th (R–S): if $G = M_1 \cup \cdots \cup M_r$, M_i induced of size t, then $t = o(N)$ (Hard) $\Rightarrow$ Roth Th

bound on r as not as good as in NT

Construction: (Behrend) $\exists A \subseteq [N]$, A has no 3-AP

$$|A| \geq \frac{N}{e^{c\sqrt{\log N}}}$$

Cor: $\exists G = M_1 \cup \cdots \cup M_r$ s.t. $|M_i| = n^{1-o(1)}$, M_i induced, $r = cN$

yet \# of edges $o(N^2)$

(surprise: large induced matching and rather dense graph)

ONE OF Main RESULTS

Th: Alon Moitra S. $\exists$ graph G on N vertices, $G = M_1 \cup \cdots \cup M_t$

- $M_i \equiv$ induced maching of size $N^{1-o(1)}$
- \# of edges in G $= (1-o(1))\binom{N}{2}$

(moreover if $|M_i| = N^{1-\varepsilon}$ then we can cover all but $N^{2-\delta}$ edges)

Shared communication channels

Model B-L-M 90's: N stations, each has one transmitter and s receivers.

GOAL: use few rounds to exchange messages between all stations

Abstruction: $K_{N,N} \equiv$ complete bipartite graph

split $K_{N,N} = G_1 \cup G_2 \cup \cdots \cup G_s$ G_i corresponds to reciver i

Round: every G_i transmits message along induced matching

\# rounds $\equiv$ \# of induced matching in decomposition of G_i

Explanation: At every round some subset of vertices on the left chooses to transmit using G_i (reciver i) vertex send message to all it neighbors on the right. If vertex on the righ gets in its i-receiver ≥ 2 messages they are corrupted. So vertex recives message only if it is unique message reaching it. So messages which were transmitted successfully form induced match.

EASY: Every round send one message $\Rightarrow N^2$ rounds

: B-L-M using s-recivers $\Rightarrow \frac{N^2}{(\log N)^{s-1}}$ rounds

Th A-M-S: $\forall \varepsilon$ $\exists$ const $s(\varepsilon)$ and protocol with s recivers which terminates in $N^{1+\varepsilon}$ rounds

Pf. split $K_{N,N} = G_1 \cup \cdots \cup G_s$ s.t. each G_i is union of $N^{1+\varepsilon}$ induced matchings of size $N^{1-\varepsilon}$

CONSTRUCTION:

Vertices $V \in [c]^n$ $\quad [c] = \{1,2,\dots,c\}$ even coordinates

Def: Pick even $x, y \in [c]^n$ $\quad$ Let $\mu = \mathbb{E}\left[\,|x-y|^2\right]$ typical distance

Edges of G $\quad x \sim y \quad$ if $\quad |x-y|^2 = (1+o(1))\mu$

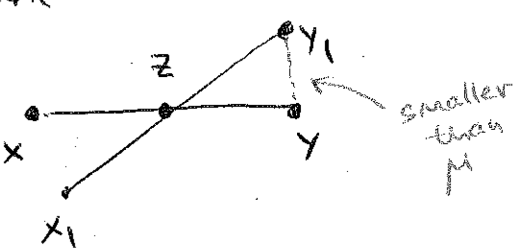
L: Since $|x-y|^2 = \Sigma (x_i - y_i)^2$ is sum of i.i.d rand. var.
most pairs are at typical distance
so # of edges in G $= (1-o(1))\binom{N}{2}$

(cheating!!!)
Matchings $\forall\, z \in [c]^n$. Let

$$V_z = \{\, x \in V(G) \mid |x-z|^2 = (1-o(1))\tfrac{\mu}{4} \,\}$$

Look on edges of G span by set V_z (both endpoints in V_z)

z is middle point



$|x-y|^2 \approx \mu \quad |x_1-y_1|^2 \approx \mu$
$|x-z|^2,\ |y-z|^2 \approx \frac{\mu}{4}$
$|x_1-z|^2,\ |y_1-z|^2 \approx \mu/4$

Cor: (i) These edges form matching
(ii) every edge of G belongs to one of such matchings
(iii) matching induced since distance (y_1, y), (x_1, x) too short