# Independence Polynomials of $k$-trees

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12 June, 2013

## Definition

**Independence Polynomial of $G$**

Prodinger and Tichy, 1982

Gutman and Harary, 1983

Let G be a graph. For $s \geq 0$ let $f_s = f_s(G)$ be the number of independent vertex sets of cardinality $s$ in G. Then the independence polynomial of $G$, denoted "$G(x)$", is defined by

$$G(x) = \sum_{s \geq 0} f_s x^s$$

**Properties of Coefficients of Independence Polynomials**

*Let $G$ be a graph with $n$ vertices and $m$ edges. Then:*

i) $f_0 = 1$;

ii) $f_1 = n$;

iii) $f_2 = \binom{n}{2} - m$;

iv) *For $s \geq 3, f_s$ is not determined by $n$ and $m$:*

$$P_4(x) = 1 + 4x + 3x^2$$

$$St_4^1(x) = 1 + 4x + 3x^2 + x^3.$$

**Examples of Independence Polynomials**

$$K_n(x) = 1 + nx$$

$$E_n(x) = (1 + x)^n = \sum_{s \geq 0} \binom{n}{s} x^s$$

$$St_n^1(x) = x + (1 + x)^{n-1} = 1 + nx + \sum_{s \geq 2} \binom{n-1}{s} x^s$$

# Reduction Formulas for Independence Polynomials

i) $(G \cup H)(x) = G(x)H(x)$;

ii) $(G \oplus H)(x) = -1 + G(x) + H(x)$;

**Vertex Reduction**

iii) *Let $v$ be a vertex of $G$. Let $G_1 = G - v$ and let $G_2 = G - N[v]$. Then $G(x) = G_1(x) + xG_2(x)$.*

**Coefficient Version of Vertex Reduction**

*Let $v$ be a vertex of $G$.*

$$f_s(G) = f_s(G_1) + f_{s-1}(G_2) \qquad (s \geq 1)$$

**Edge Reduction**

iv) *Let $e$ be an edge of $G$. Let $G_3 = G - e$ and $G_4 = G - (N[v] \cup N[w])$. Then $G(x) = G_3(x) - x^2 G_4(x)$.*

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$$P_n(x) = c_1(x)\left(\frac{1+\sqrt{1+4x}}{2}\right)^n + c_2(x)\left(\frac{1-\sqrt{1+4x}}{2}\right)^n$$

Use initial conditions: $P_0(x) = 1$ and $P_1(x) = 1 + x$.

Solving for $c_1(x)$ and $c_2(x)$ and simplifying yields:

$$P_n(x) = \frac{1}{\sqrt{1+4x}}\left[\left(\frac{1+\sqrt{1+4x}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{1+4x}}{2}\right)^{n+2}\right]$$

*Similarly:* $f_s(P_n) = \binom{n+1-s}{s}$

## Theorem

*Wingard, 1995*

*Let $T$ be a tree with $n$ vertices with* $T(x) = \sum_{s \geq 0} f_s x^s$.

*Then for $s \neq 1$:*

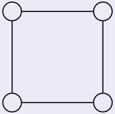
$$\binom{n+1-s}{s} \leq f_s \leq \binom{n-1}{s}$$

*Wingard's bounds are sharp*

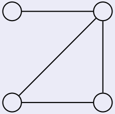
$f_s(P_n) = \binom{n+1-s}{s}$ *and if* $s \neq 1$ $f_s(St_n^1) = \binom{n-1}{s}$

# A pair of non-isomorphic graphs with the same independence polynomial:

$G_1$



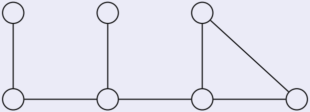
$G_2$



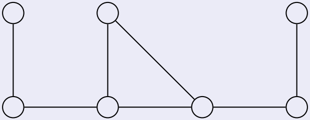
$$G_1(x) = 1 + 4x + 2x^2 = G_2(x)$$

# Another pair:

$G_3$



$G_4$



$$G_3(x) = 1 + 7x + 14x^2 + 8x^3 = G_4(x)$$

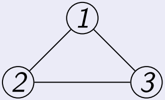
# $k$-trees

*Definition:*
*(Beineke and Pippert, 1969)*

(1) $K_{k+1}$ *is a* $k$*-tree on* $k+1$ *vertices;*

(2) *If* $T$ *is a* $k$*-tree on* $n$ *vertices and* $C$ *is a* $k$*-clique of* $T$*, then a* $k$*-tree on* $n+1$ *vertices is formed by adjoining a new vertex* $v$ *to* $T$ *and joining* $v$ *by an edge to each vertex of* $C$*.*

# Building a 2-tree



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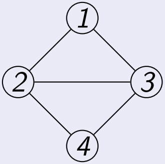

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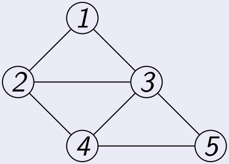

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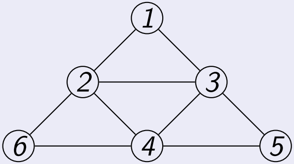

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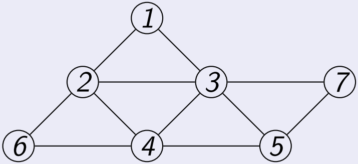

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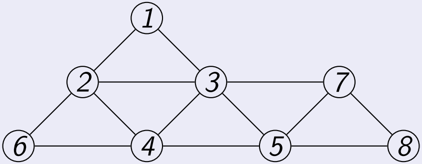
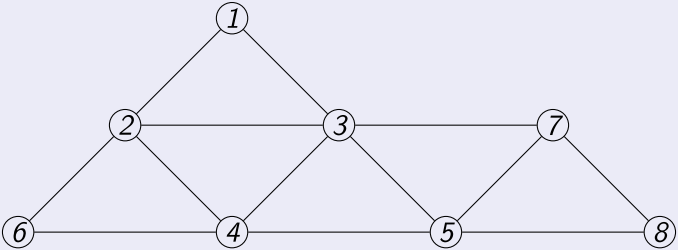


## Definition

Let $T$ be a $k$-tree and $v$ a vertex of $T$. If the neighbors of $v$ induce a clique, then $v$ is said to be a **simplicial vertex**.




*Vertices* 1, 6, *and* 8 *are simplicial.*

*In a $k$-tree, a vertex $v$ is simplicial if and only if degree$(v) = k$.*

# Some facts about $k$-trees:

*Let $T$ be a $k$-**tree** on $n$ vertices, $n \geq k + 2$. Then:*

(0) $\delta(T) = k$;

(1) *$T$ is maximally $k$-degenerate;*

(2) *$T$ has at least two simplicial vertices;*

(3) $\omega(T) = k + 1$;

(4) $\chi(T) = k + 1$;

(5) *$T$ is uniquely $(k + 1)$-colorable;*

(6) $\chi_T(\lambda)=\lambda^{\underline{k}}(\lambda - k)^{n-k}$;

(7) *$T$ has exactly $n - k$ $(k + 1)$-cliques and exactly $kn - \binom{k+1}{2}$ edges.*

# Examples of $k$-trees

## Definition

$P_n^k$: **the $k$-path on n vertices** with $n \geq k+1$.

Let $v_1, v_2, \ldots v_{k+1}$ be a $(k+1)$-clique.

For $k+2 \leq i \leq n$, let $v_i$ be adjacent to $v_{i-1}, v_{i-2}, \ldots v_{i-k}$.

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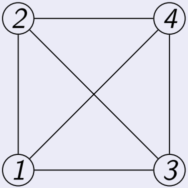

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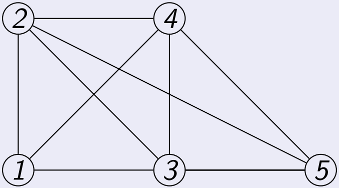

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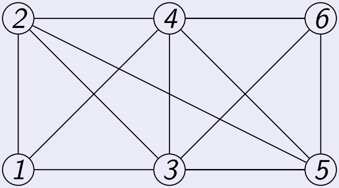

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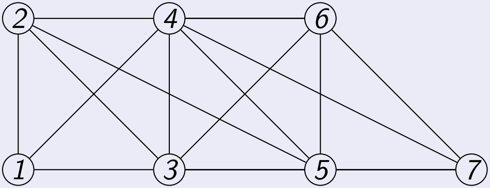

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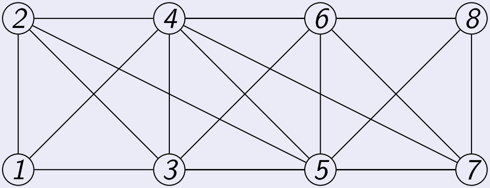

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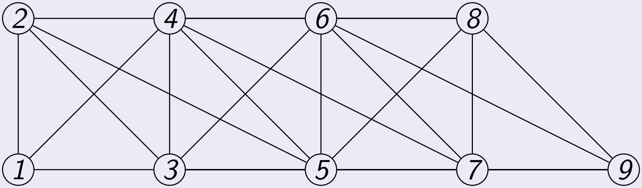

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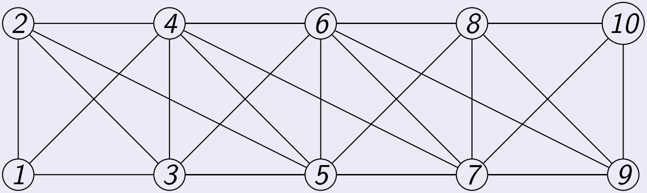

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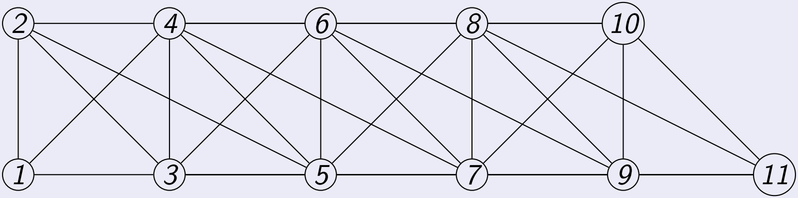

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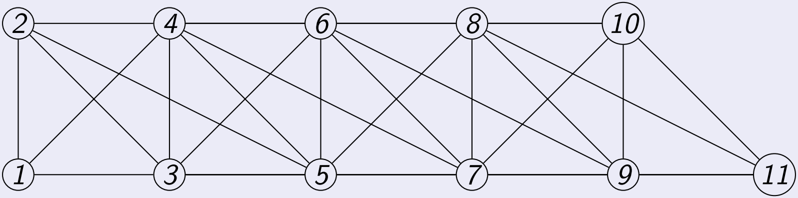
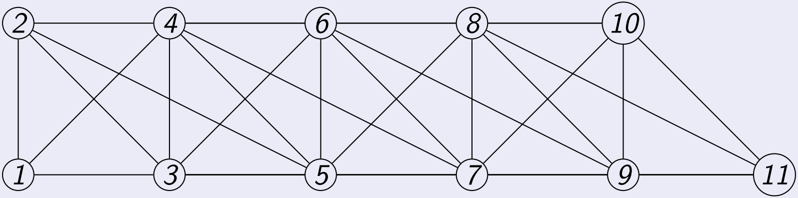


## $P_{11}^3$: the 3-path on eleven vertices



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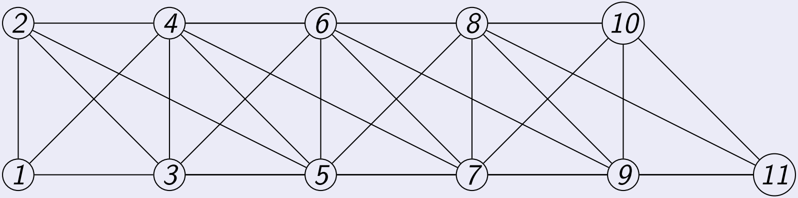
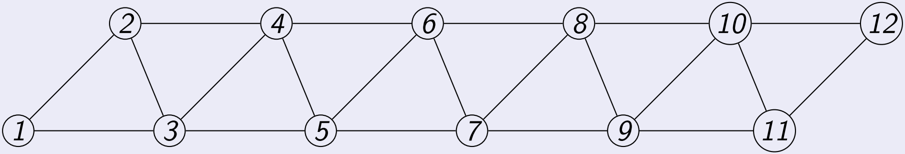



## $P^2_{12}$: the 2-path on twelve vertices

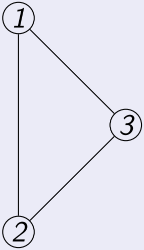


## Definition

$St_n^k$**: the k-star on** $n$ **vertices with** $n \geq k+1$.

1. Let $v_1, v_2, \ldots v_k$ be a $k$-clique.
2. For $k+1 \leq i \leq n$, let $v_i$ be adjacent to $v_1, v_2, \ldots v_k$.

### $St_8^2$ :the 2-star on eight vertices



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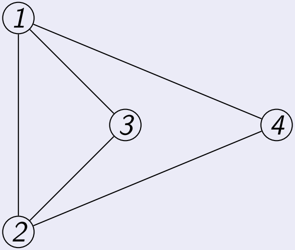

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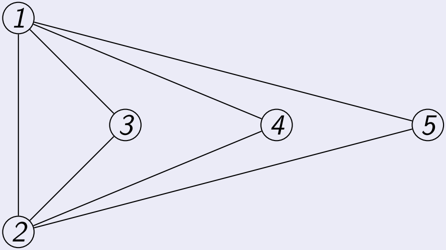

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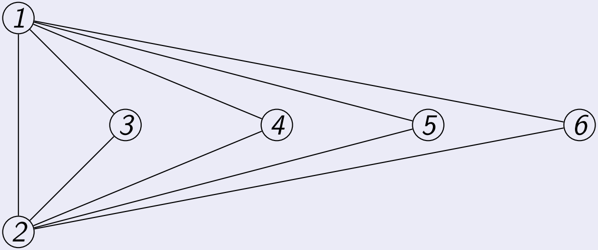

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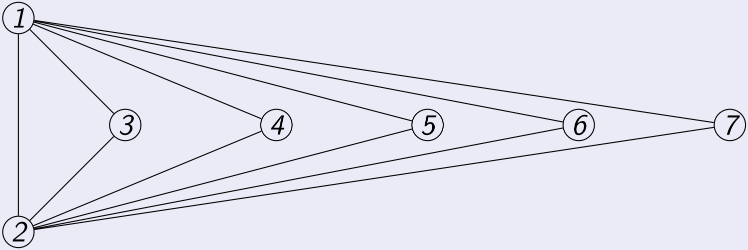

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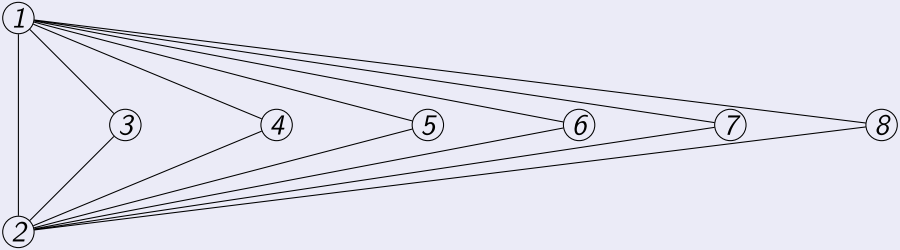
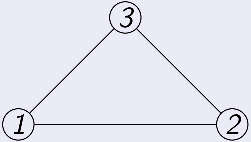


## Definition

$Sp_n^k$**: the** $k$**-spiral on n vertices with** $n \geq k+2$**.**

1. Let $v_1, v_2, \ldots v_{k+1}$ be a $k+1$-clique.
2. For $k+2 \leq i \leq n$, let $v_i$ be adjacent to $v_2, v_3, \ldots v_k$ and $v_{i-1}$.

## $Sp_7^2$ :the 2-spiral on seven vertices



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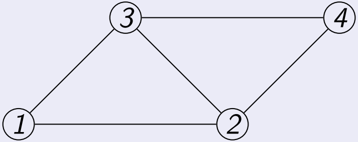

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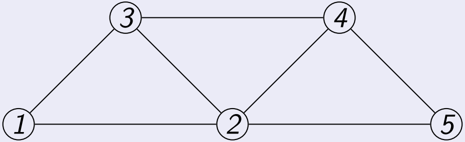

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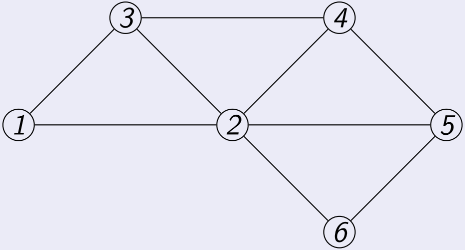

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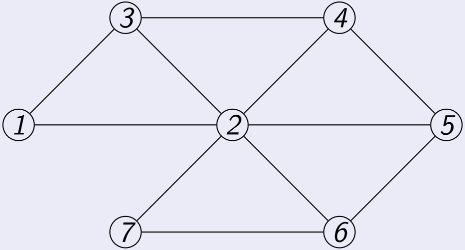

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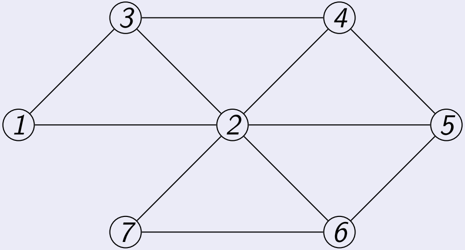

# Independence Polynomials of Some $k$-trees

Song, Wei et al. 2010

i) $St_n^k(x) = kx + (1+x)^{n-k} = 1 + nx + \sum_{s \geq 2} \binom{n-k}{s} x^s$

ii) $Sp_n^k(x) = (k-1)x + P_{n+1-k}(x) = 1 + nx + \sum_{s \geq 2} \binom{n+2-k-s}{s} x^s$

iii) $P_n^k(x) = \sum_{s \geq 0} \binom{n+k-ks}{s} x^s$

# Generalizing Wingard's Inequality

**Theorem**

*Song, Wei et al. 2010*

*Let $G$ be a $k$-degenerate graph with $n$ vertices. Then, for $s \neq 1$:*

i) $\binom{n+k-ks}{s} \leq f_s(G)$;

ii) *If $G$ is maximal $k$-degenerate, then* $f_s(G) \leq \binom{n-k}{s}$.

*Corollary*

*Let $T$ be a $k$-tree with $n$ vertices. Then, for $s \neq 1$:*

$$\binom{n+k-ks}{s} \leq f_s(G) \leq \binom{n-k}{s}$$

*A construction of well-covered graphs:*

*Let $G$ be a graph and $C_1, C_2, \ldots, C_r$ a collection of cliques partitioning the vertex set of $G$. For $1 \leq i \leq r$ let $H_i$ be a clique and join each vertex of $C_i$ to each vertex of $H_i$. The resulting graph is called a* **Corona** *over G.*

*A construction of well-covered graphs:*

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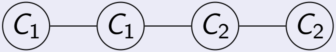
*A corona over $P_4$*



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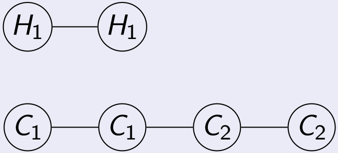

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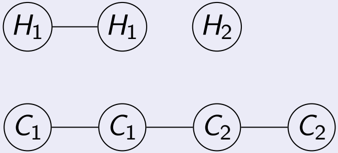

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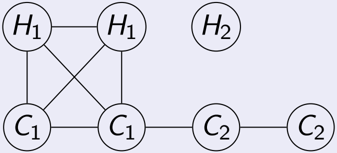

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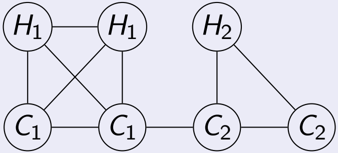


*Independence Polynomials of Coronas*

**Theorem**

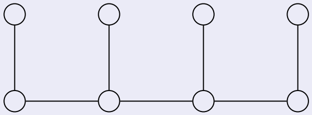
*Gutman, 1992, Song, Wei et al. 2012*

*Let $G$ be a graph. Let $\{C_1, C_2, \ldots C_r\}$ be cliques partitioning the vertex set of $G$ and let $H_i = K_s$ for $1 \leq i \leq r$. If $J$ is the corona over $G$, then*

$$J(x) = (1 + sx)^r G(\frac{x}{1+sx})$$

# Example

Let $G$ be $P_4$, each $C_i$ and each $H_i$ a singleton. The corona $J$ is a "comb."



$P_4(x) = 1 + 4x + 3x^2$
$r = 4$
$s = 1$

$$J(x) = [1 + x]^4[1 + 4(\frac{x}{1+x}) + 3(\frac{x}{1+x})^2]$$
$$= 1 + 8x + 21x^2 + 22x^3 + 8x^4$$

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**¡MUCHAS GRACIAS, AMIGOS!**