Stable-Π Partitions of Graphs

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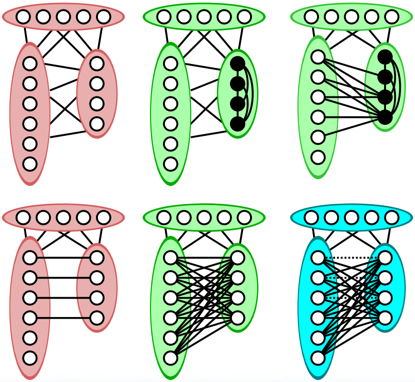
Vertex partitions of graphs

Graph = simple (undirected, no loops, no parallel edges)

Vertex partition of a graph G = partition of $V(G)$ into $V_1 \cup \ldots \cup V_t$



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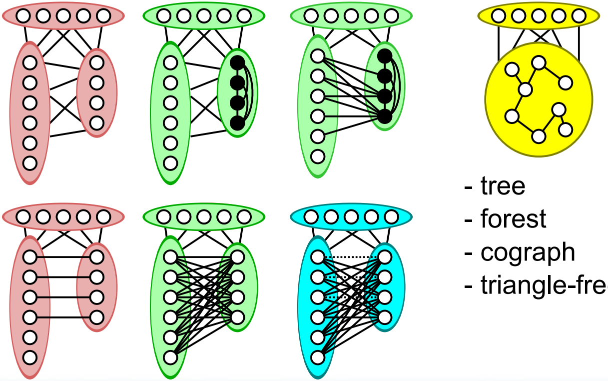
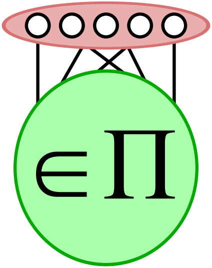


- tree
- forest
- cograph
- triangle-free graph

STABLE-Π partition

$\Pi =$ set of graphs, closed under isomorphism (a.k.a. graph *property*)

Vertex partition of G into an independent set and a graph in Π



STABLE-Π partition problem

Given G find an independent set S such that $G - S \in \Pi$

Hereditary properties

Property Π = isomorphism-closed set of graphs

Hereditary property = closed under vertex removal

hereditary: planar, perfect, chordal

non-hereditary: cycles, trees, k-trees, k-connected graphs

Speed = growth rate of Π_n

... where Π_n denotes the graphs in Π with vertex set $\{1, \ldots, n\}$

[Alekseev'93, Scheinerman-Zito'94, Balogh-Bollobás-Weinreich'00]

Speed of hereditary properties

Theorem [Alekseev'93, Bollobás-Thomason'95]: If Π is hereditary,

$$\log |\Pi_n| = \left(1 - \frac{1}{c(\Pi)}\right)\frac{n^2}{2} + o(n^2)$$

where $c(\Pi)$ is integer

Low-speed $c(\Pi) = 1$: [Alekseev'97, Balogh-Bollobás-Weinreich'00]

constant $|\Pi_n| \leq 2$

polynomial $|\Pi_n| = n^{\Theta(1)}$

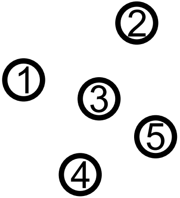
exponential $|\Pi_n| = \Theta(1)^n$

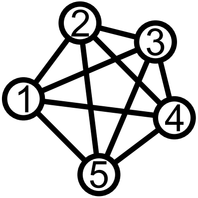
factorial $|\Pi_n| = n^{\Theta(1)n}$

. . . *subfactorial* = all layers below factorial

Constant layer

constant $|\Pi_n| \leq 2$





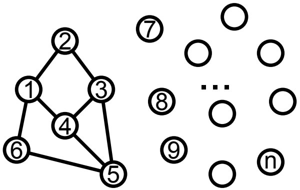
Polynomial layer

…once Π_n contains a graph with both an edge and a non-edge

polynomial $|\Pi_n| = n^{\Theta(1)}$

graph on k vertices

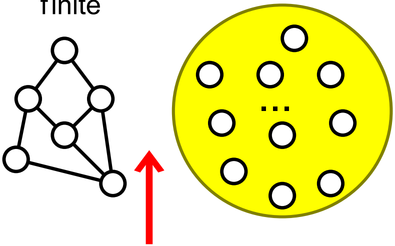
(n-k) isolated vertices



at most n^k labelled graphs

finite

clique or independent set



all or no edges

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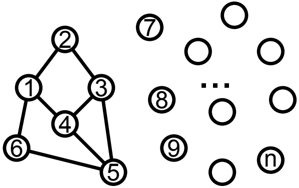
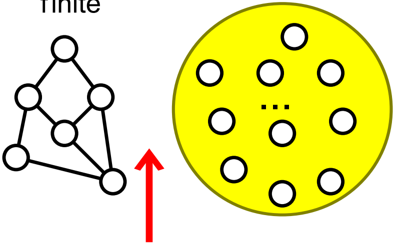


finite

clique or independent set



at most n^k labelled graphs

all or no edges

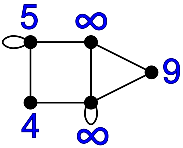
+ **repeat** for finite number of finite graphs $H_1, \ldots, H_t$

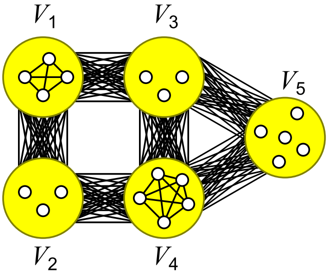
Exponential layer

…once Π_n contains a graph with $2+$ **twin** classes of unbounded size

exponential $|\Pi_n| = \Theta(1)^n$

H
(k vertices)





$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$
(k x k)

$b = (5,4,\infty,\infty,9)$

$|V_i| \leq b(i)$

at most $\sum_{n_1+\ldots+n_k=n} \binom{n}{n_1\, n_2 \ldots n_k} = k^n$ graphs

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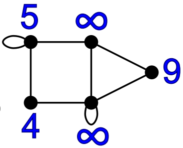

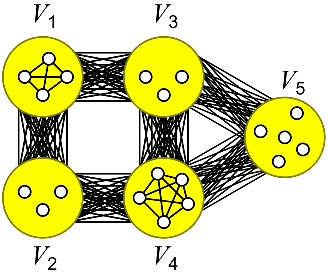

P(M,b)

denotes the set of graphs constructed this way

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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$P(M,b)$

$$\underset{(k \times k)}{M} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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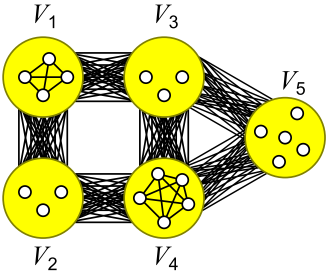


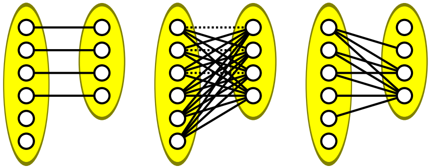
$\exists M_1, M_2, \ldots, M_t$ and $b_1, b_2, \ldots, b_t$:

$$\Pi = \bigcup_{i=1}^{t} P(M_i, b_i)$$

Factorial layer

…once Π_n contains a graph with unbounded number of **twin** classes

factorial $|\Pi_n| = n^{\Theta(1)n}$



+**split** and **co-bipartite** equivalents (independent sets into cliques)

max-degree 1, bipartite almost complete, chain graphs (bip. $2K_2$-free)

…these are exactly all "minimal" factorial classes [A97,BBW00]

Theorem

Π is (at least) factorial $\iff$ Π contains at least one of the 9 classes

Minimal factorial classes

why are minimal classes useful? could help us understand the structure of all factorial classes

Problem

Given a bipartite G find a largest subgraph of G that is Π

[Yannakkakis'81] polytime if and only if Π is **subfactorial**
(or all bipartite graphs)

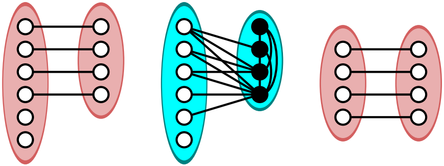
hard for minimal factorial classes $\Rightarrow$ *hard* for all factorial classes

Results

STABLE-Π: Given G find an independent set S such that $G - S \in \Pi$

- polytime solvable for all subfactorial properties
- polytime solvable for 7 of 9 minimal factorial properties
- NP-hard for strict cases

Related work:



[Brandstädt-Le-Szymczak'98] *Max-degree 1*
[Mahadev-Peled'95, Le-Le'02] *Threshold graphs*
[Grinstead-Slater-Sherwani-Holmes'93] *"Efficient edge domination"*

Result #1 - subfactorial classes

Theorem

STABLE-Π is polytime for every subfactorial Π

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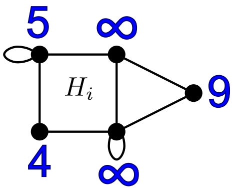
Theorem

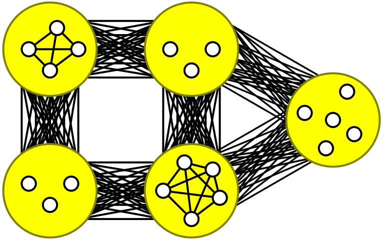
STABLE-Π is polytime for every subfactorial Π

structural theorem for Π [A97,BBW00]

$\exists H_1, \ldots, H_k$ and $b_1, \ldots, b_k$:



$b_i = (5, 4, \infty, \infty, 9)$



Result #1 - subfactorial classes

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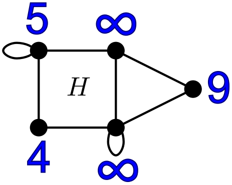

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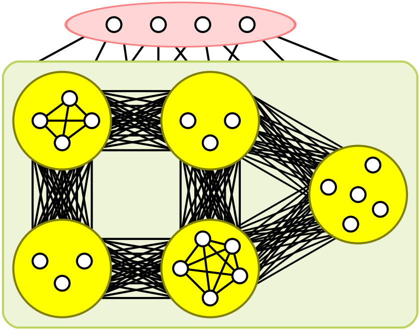
G $\longrightarrow$



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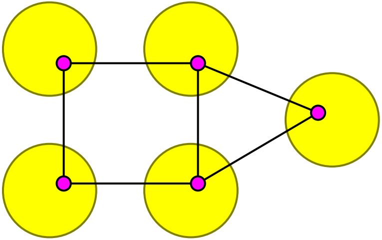

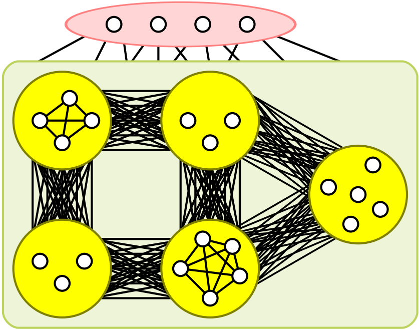

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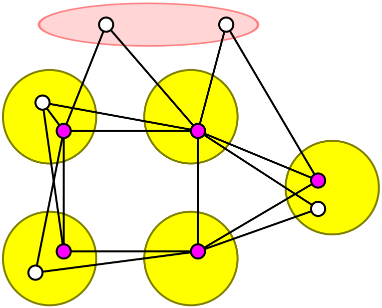

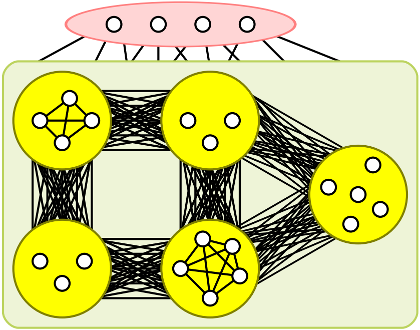

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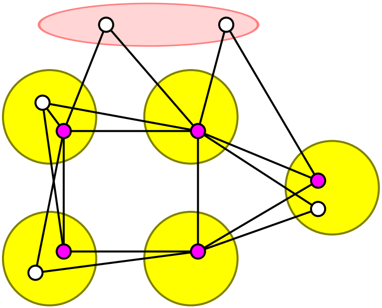

complexity:

- guess the representatives
- guess all **finite-size** V_i's
  (those with $b(i) < \infty$)

$(n+1)^c$ where $c = |H| + \sum_{b_i < \infty} b_i$

2SAT $O(n^2)$

Result #2 - Minimal factorial classes

Theorem

STABLE-Π is polytime for 7 of 9 minimal factorial classes

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Theorem

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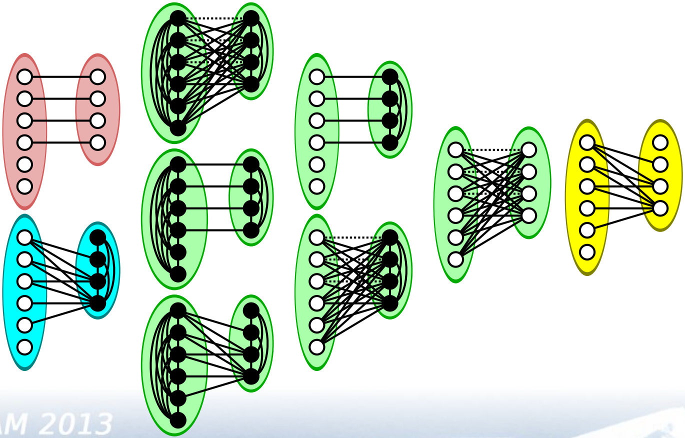
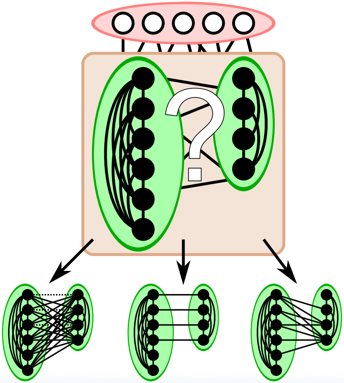
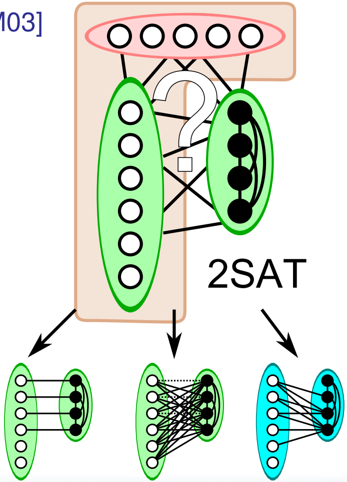


Proof techniques

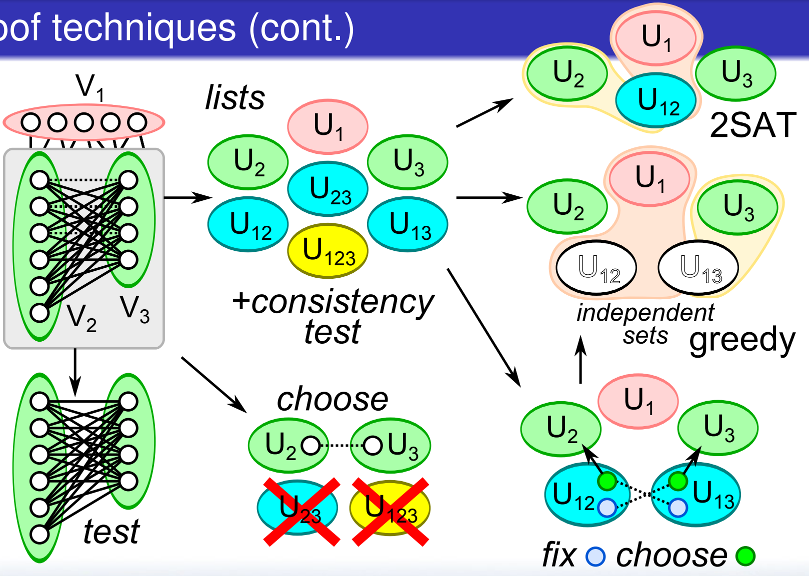
Sparse-dense partitions [AFL04,FHKM03]

at most n^4 partitions



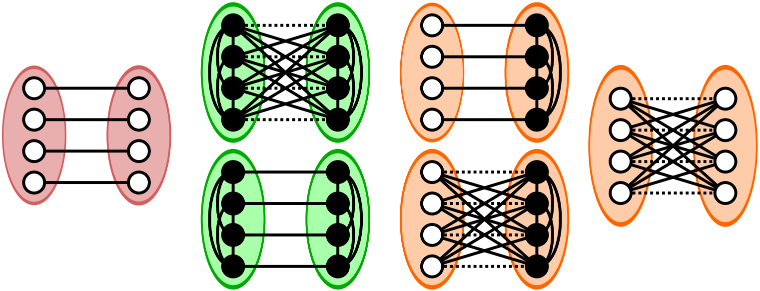


Proof techniques (cont.)



Result #3 - Strict cases

“strict” variants of the minimal factorial classes



[Grinstead-Slater-Sherwani-Holmes’93] “Efficient edge domination”

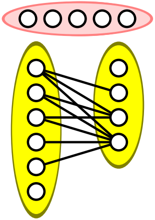
. . . shows a narrow dividing line between easy and hard

Summary

STABLE-Π problem for properties Π of low speed (*subfactorial*, *minimal factorial*)

Open problems:

- solve the remaining minimal case
- where is the dividing line?
- monotonicity?
- structure of factorial classes



Thank you for your attention!