Cyclic, Simple and Indecomposable Three-Fold Triple Systems

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Outline

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Definitions and examples

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$V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

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$D = \{1, 2, 3, 4\} \Rightarrow \frac{4}{1}\frac{2}{2}\frac{3}{3}\frac{2}{4}\frac{4}{5}\frac{3}{6}\frac{1}{7}\frac{1}{8}$ (Skolem sequence of order 4)

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$D = \{1, 2\} \Rightarrow (1, 1, 2, 2, 2, 2, 1, 1)$ (two-fold Skolem sequence of order 2)

Rees and Shalaby (2000)

Remark

In 2000, Rees and Shalaby showed using Skolem-type sequences that there exists cyclic, simple and indecomposable two-fold triple systems for all admissible orders.

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Example

2-fold S_3 : $\frac{3}{1}\frac{1}{2}\frac{1}{3}\frac{3}{4}\frac{3}{5}\frac{1}{6}\frac{1}{7}\frac{3}{8}\frac{2}{9}\frac{2}{10}\frac{2}{11}\frac{2}{12}$

$\Rightarrow \{(6,7),(2,3),(9,11),(10,12),(1,4),(5,8)\}$

$\Rightarrow \{0,1,10\}\{0,1,6\}\{0,2,14\}\{0,2,15\}\{0,3,7\}\{0,3,11\}(\text{mod } 19)$

Kramer (1974)

Theorem

The blocks $\{0, \alpha, -\alpha\} (\text{mod } v) | \alpha = 0, 1, \ldots, \frac{1}{2}(v-1)$ *for* $v \equiv 1 \text{ or } 5 \pmod 6$ *form an indecomposable three-fold triple system.*

Cyclic Designs with $k = 3$ and $\lambda = 3$

Construction (D. Silvesan and N. Shalaby, 2012)

Let $S_n = (s_1, s_2, \ldots, s_{2n})$ be a Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in S_n for which $b_i - a_i = i$. Then the triples $\{\{0, i, b_i\} | i = 1, \ldots, n\} (\text{mod } 2n + 1)$ form a cyclic $\text{TS}_3(2n + 1)$.

Example

$S_4 = (1, 1, 3, 4, 2, 3, 2, 4) \Rightarrow (1, 2), (5, 7), (3, 6), (4, 8) \Rightarrow$ cyclic $\text{TS}_3(9) : \{0, 1, 2\}\{0, 2, 7\}\{0, 3, 6\}\{0, 4, 8\} (\text{mod } 9)$

Cyclic Designs with $k = 3$ and $\lambda = 3$

Construction (D. Silvesan and N. Shalaby, 2012)

Let $hS_n = (s_1, s_2, \ldots, s_{2n-1}, s_{2n+1})$ be a hooked Skolem sequence of order n and let $\{(a_i, b_i) | 1 \leq i \leq n\}$ be the pairs of positions in hS_n for which $b_i - a_i = i$. Then the triples $\{\{0, i, b_i + 1\} | i = 1, \ldots, n\} (\text{mod } 2n + 1)$ form the base blocks for a cyclic block design $\text{TS}_3(2n + 1)$.

Example

$hS_3 = (1, 1, 2, 3, 2, 0, 3) \Rightarrow (1, 2), (3, 5), (4, 7) \Rightarrow$ cyclic $\text{TS}_3(7) : \{0, 1, 3\}\{0, 2, 6\}\{0, 3, 1\} (\text{mod } 7)$

Lemma

For every $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$, *there is a Skolem sequuence of order* n *starting with a* 1 *and ending with a* 2.

Lemma

For every $n \equiv 2$ or $3 \pmod 4$, $n \geq 7$, *there is a hooked Skolem sequuence of order* n *starting with a* 1 *and ending with a* 2.

Theorem

There exists simple three-fold cyclic triple systems $TS_3(v)$ for all $v = 2n + 1,\ v \geq 15$.

Proof:

Let $v = 2n + 1$, $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$. Apply the first construction to a Skolem sequence of order n starting with a 1 and ending with a 2.

Suppose that the construction above produces $\{x, y, z\}$ as a repeated block. With regards to the first construction, any block $\{x, y, z\}$ is of the form $\{0, i, b_i\} + k$ for some $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}_{2n+1}$. Hence if $\{x, y, z\}$ is a repeated block we have

$$\{0, i_1, b_{i_1}\} + k_1 = \{0, i_2, b_{i_2}\} + k_2$$

whence

$$\{0, i_2, b_{i_2}\} = \{0, i_1, b_{i_1}\} + k$$

for some $i_1, i_2 \in \{1, 2, \ldots, n\}$ and some $k \in \mathbb{Z}_{2n+1}$.

Proof:

If $k = 0$ we have $i_2 = b_{i_1}$ and $i_1 = b_{i_2}$ which is impossible since $b_{i_1} \geq i_1 + 1$ and $b_{i_2} \geq i_2 + 1$ from the definition of a Skolem sequence.

If $k = i_2$ we have $\begin{cases} i_1 + i_2 = 2n + 1 \\ b_{i_1} + i_2 = b_{i_2} \end{cases}$ or $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n + 1. \end{cases}$

Since both i_1 and i_2 are at most n, it is impossible to have $i_1 + i_2 = 2n + 1$.

If $k = b_{i_2}$ we have

$$\begin{cases} i_1 + b_{i_2} = 2n + 1 \\ b_{i_1} + b_{i_2} = i_2 + 2n + 1 \end{cases} \Leftrightarrow \begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n + 1 \end{cases} \text{ or}$$

$$\begin{cases} i_1 + b_{i_2} = i_2 \\ b_{i_1} + b_{i_2} = 2n + 1. \end{cases}$$

Since $b_{i_2} > i_2$, it is impossible to have $i_1 + b_{i_2} = i_2$.

Proof:

So, to prove that a system has no repeated blocks is enough to show that $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n+1 \end{cases}$ or $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n+1 \end{cases}$ are not satisfied.
Note also that $i_1 = \frac{v}{3}$ and $b_{i_1} = 2\frac{v}{3}$ is not allowed.

Indecomposable Three-fold Triple Systems

Theorem

There exists an indecomposable three-fold cyclic triple system $TS_3(v)$ for every $v \equiv 3 \pmod 6$, $v \geq 15$.

Proof:

Let $v = 2n + 1$, $n \equiv 2$ or $3 \pmod 4$, $n \geq 7$. Apply the second construction to a hooked Skolem sequence of order n starting with a 1 and ending with a 2.

Now, for an $TS_3(2n+1)$ to be decomposable, there must be a Steiner triple system $STS(2n+1)$ in the $TS_3(2n+1)$.

Proof:

If $2n+1 \equiv 3 \pmod 6$, let $\{x_i, x_j, x_k\}$ be a triple using symbols from $\mathrm{N}_{2n+1} = \{0, 1, \ldots, 2n\}$. Let $d_{ij} = \min\{|x_i - x_j|, 2n+1-|x_i - x_j|\}$ be the difference between x_i and x_j. An STS$(2n+1)$ on N_{2n+1} must have a set of triples with the property that each difference d, $1 \leq d \leq n$, occurs exactly $2n+1$ times. Assume there is an STS$(2n+1)$ inside our $\mathrm{TS}_3(2n+1)$ and let f_α be the number of triples inside the STS$(2n+1)$ which are a cyclic shift of $\{0, \alpha, b_\alpha + 1\}$.
Now it is enough to look at the first two base blocks of our $\mathrm{TS}_3(2n+1)$. These are $\{0,1,3\}(\text{mod } 2n+1)$ and $\{0,2,1\}(\text{mod } 2n+1)$. Then the existence of an STS$(2n+1)$ inside our $\mathrm{TS}_3(2n+1)$ requires that the equation $f_1 + 2f_2 = 2n+1$ must have a solution in nonnegative integers.

For $v = 15$, take the hS_7=$(1, 1, 6, 7, 3, 4, 5, 3, 6, 4, 7, 5, 2, *, 2)$.

$\{0, 1, 3\}, \{0, 1, 2\}, \{0, 3, 9\}, \{0, 4, 11\}, \{0, 5, 13\}, \{0, 6, 10\}, \{0, 7, 12\}$

$\{1, 2, 4\}, \{1, 2, 3\}$

$\{2, 3, 5\}, \{2, 3, 4\}$

$\{3, 4, 6\}, \{3, 4, 5\}$

$\{4, 5, 7\}, \{4, 5, 6\}$

$\{5, 6, 8\}, \{5, 6, 7\}$

$\{6, 7, 9\}, \{6, 7, 8\}$

$\{7, 8, 10\}, \{7, 8, 9\}$

$\{8, 9, 11\}, \{8, 9, 10\}$

$\{9, 10, 12\}, \{9, 10, 11\}$

$\{10, 11, 13\}, \{10, 11, 12\}$

$\{11, 12, 14\}, \{11, 12, 13\}$

$\{12, 13, 0\}, \{12, 13, 14\}$

$\{13, 14, 1\}, \{13, 14, 0\}$

$\{14, 0, 2\}, \{14, 0, 1\}$

Open Problems

1. Find cyclic, simple and indecomposable triple systems for $\lambda \geq 4$.

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1. Find cyclic, simple and indecomposable triple systems for $\lambda \geq 4$.
2. Find cyclically indecomposable triple systems that are decomposable for $\lambda \geq 3$.

THANK YOU!...