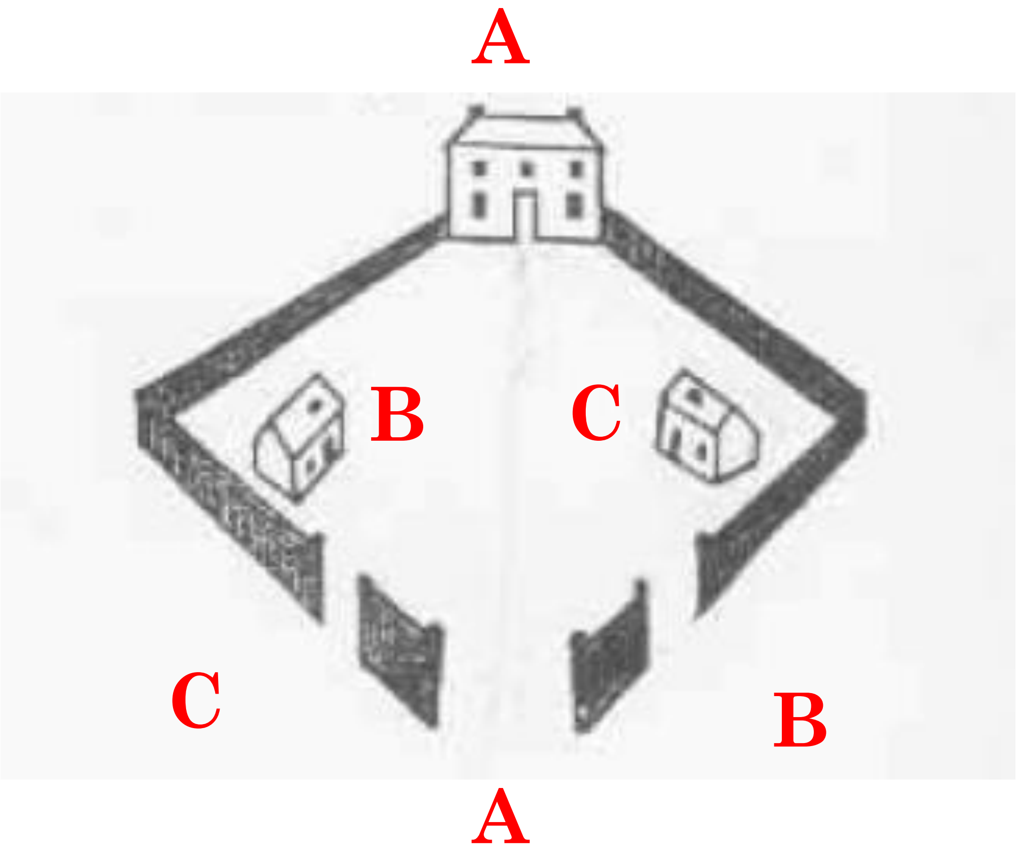
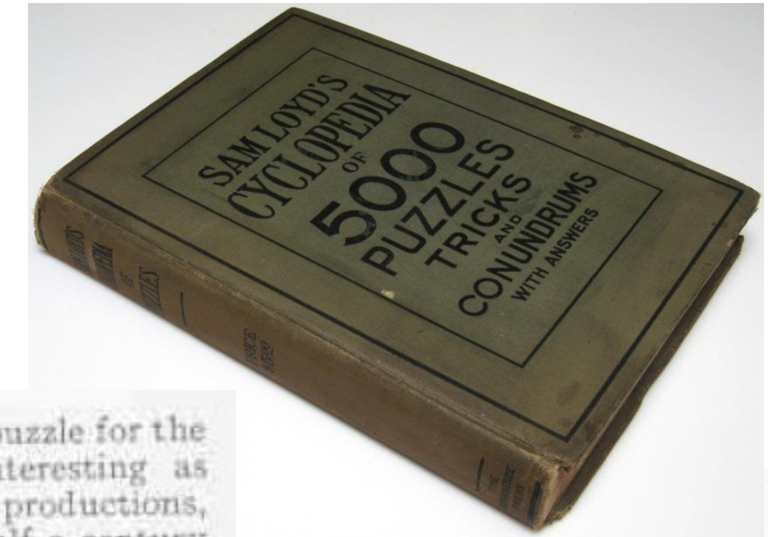


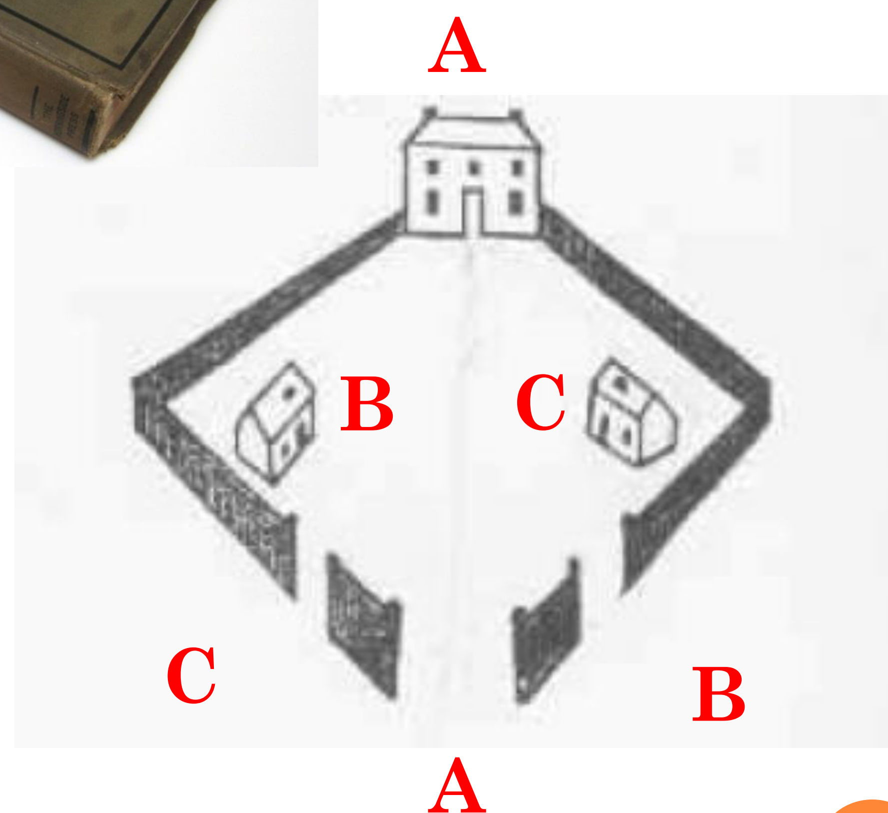
TOWARD A THEORY OF PLANARITY

AN ALGORITHM FOR SIMULTANEOUS PLANARITY?

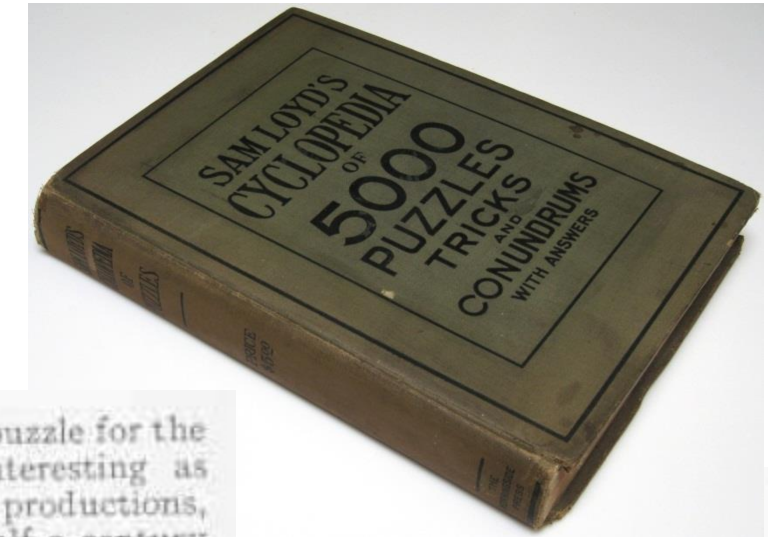
Marcus Schaefer
DePaul University



Here is an odd little puzzle for the Juveniles, which is interesting as being one of my earliest productions, published more than half a century ago. It shows the original drawing as done by a lad of nine and is given to encourage young puzzlists to attempt similar work. It is told that three neighbors, who shared a small park, as shown in the sketch, had a falling out. The owner of the large house complaining that his neighbor's chickens annoyed him, built an enclosed pathway from his door to the gate at the bottom of the picture. Then the man on the right built a path to the gate on the left, and the man on the left built a path to the gate on the right, so that none of the paths cross!



Sam Loyd's Cyclopedia of 5000 puzzles, tricks and conundrums, 1914



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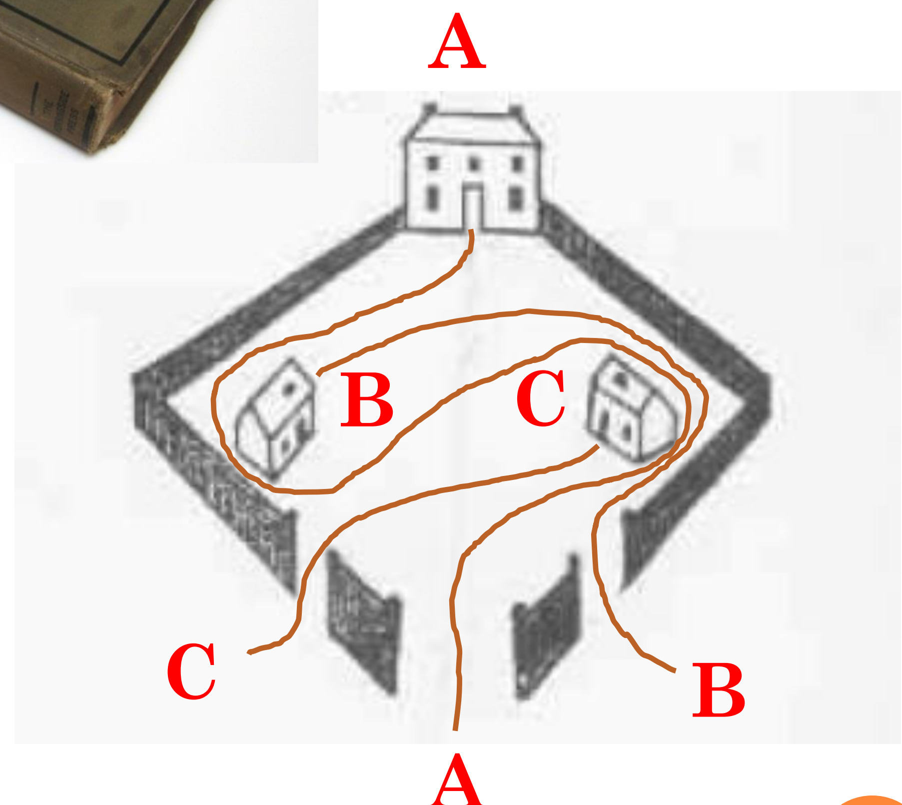

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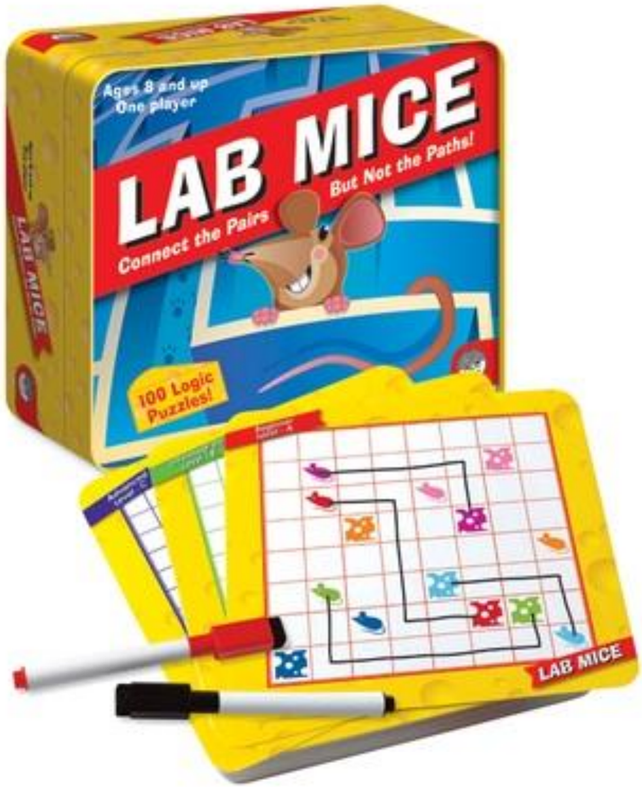
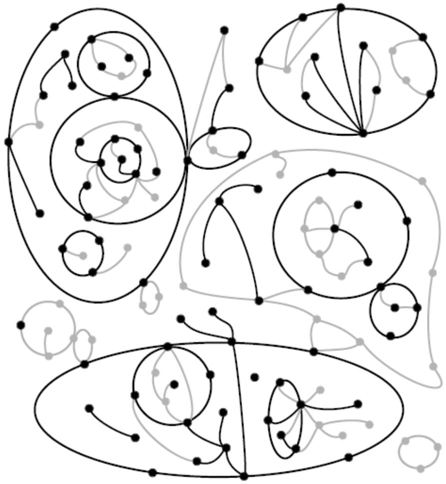

Ages 8 and up
One player
LAB MICE
Connect the Pairs
But Not the Paths!
100 Logic Puzzles!
LAB MICE

PLANARITY?

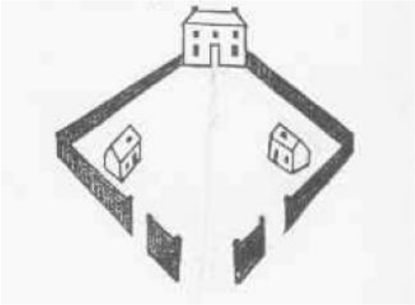
Testing Planarity of Partially Embedded Graphs *

Patrizio Angelini†, Giuseppe Di Battista†, Fabrizio Frati†, Vít Jelínek‡§, Jan Kratochvíl‡, Maurizio Patrignani†, and Ignaz Rutter♯



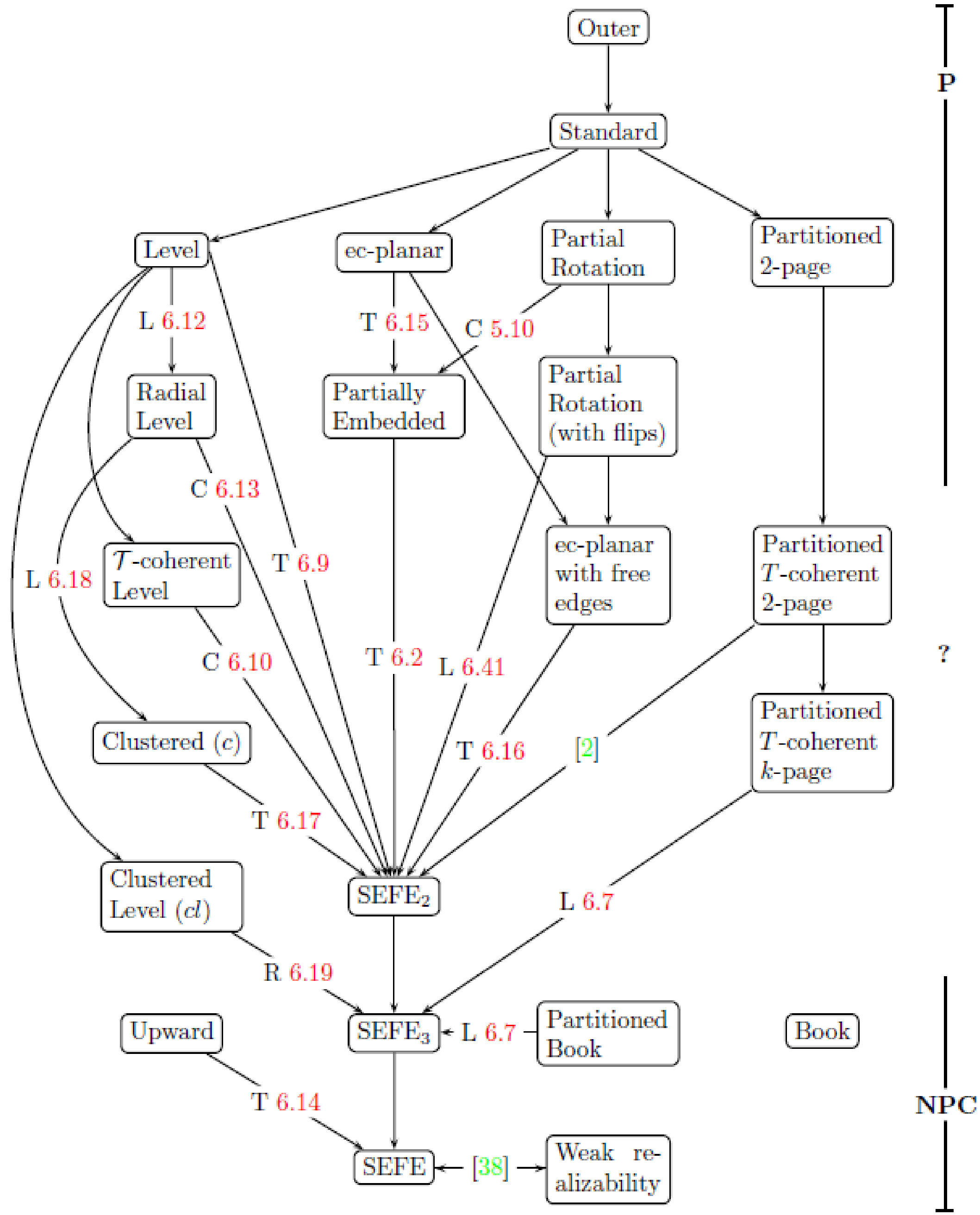
PARTIALLY EMBEDDED PLANARITY (PEP)
Input: A triplet $(G, H, \mathcal{H})$ of graphs G and H, with $H \subseteq G$, and a planar embedding $\mathcal{H}$ of H.

Question: Does G admit a planar embedding whose restriction to H is $\mathcal{H}$?



THEOREM 4.5. PEP *can be solved in linear time.*

Planarities



C-PLANARITY

Planarity for Clustered Graphs

Qing-Wen Feng   Robert F. Cohen   Peter Eades

Department of Computer Science, University of Newcastle
University Drive, Callaghan NSW 2308, AUSTRALIA
Email: qwfeng, rfc, eades@cs.newcastle.edu.au

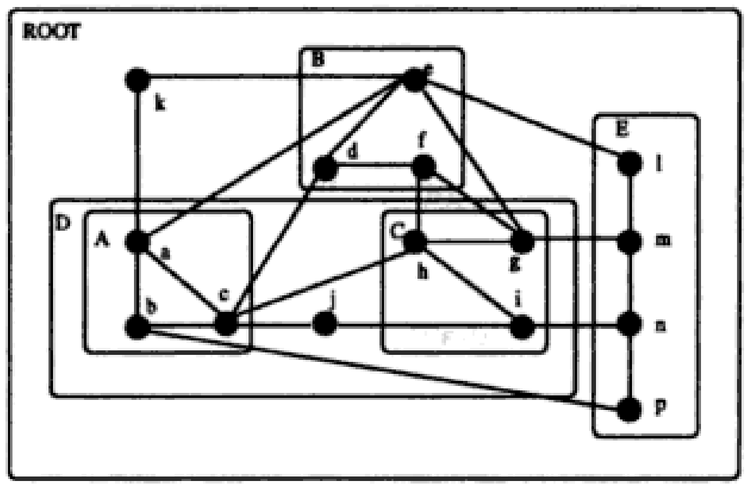
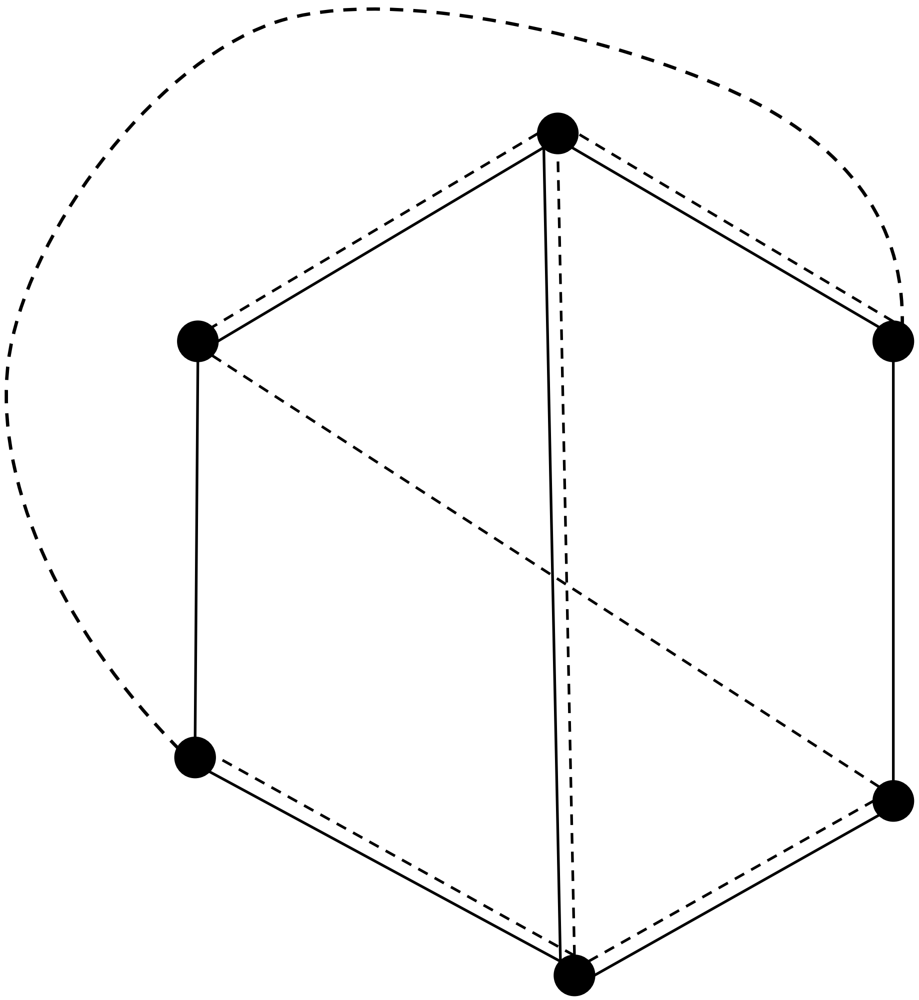


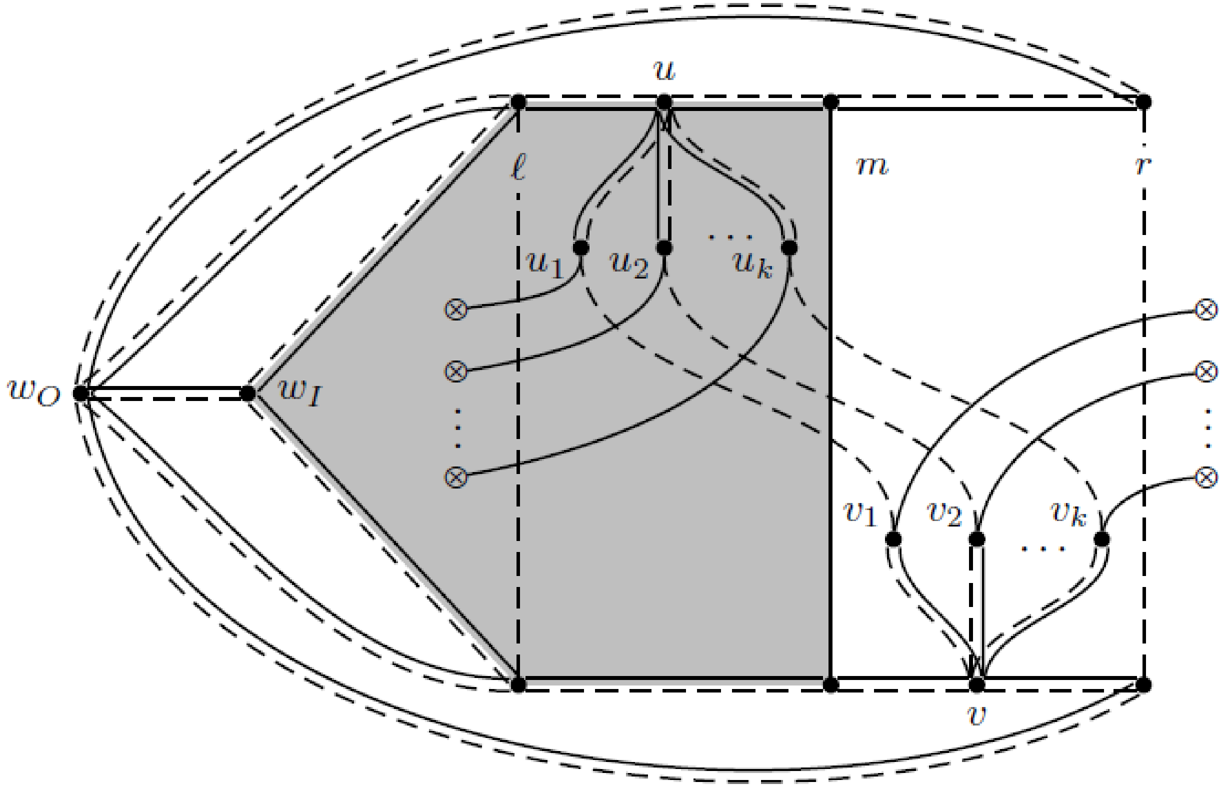
Fig. 1. An Example of a Clustered Graph

Simultaneous Embedding with Fixed Edges ($SEFE_2$)

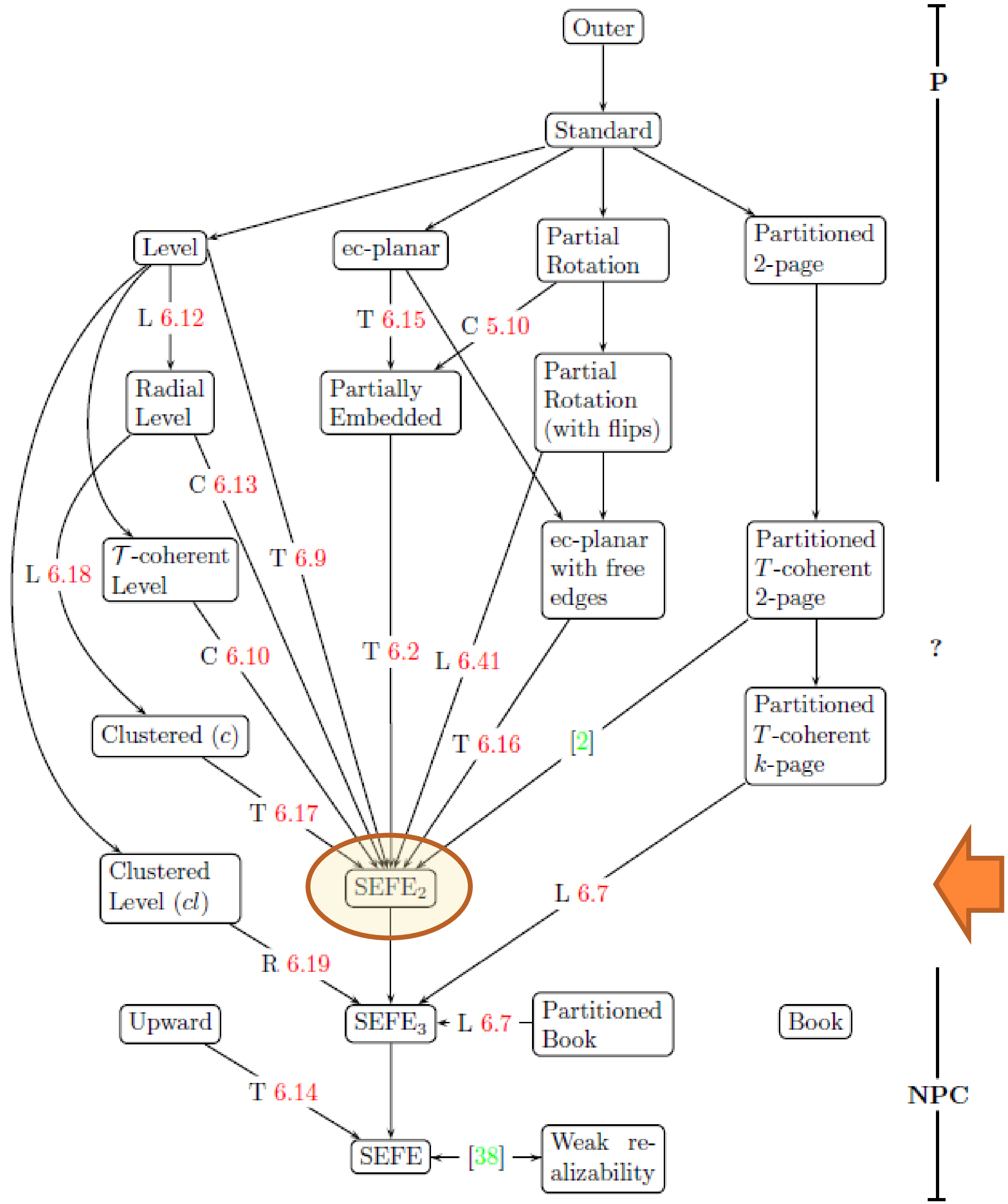


Based on example from Fowler, Jünger, Kobourov, Schulz, 2008

Reduction



PLANARITIES

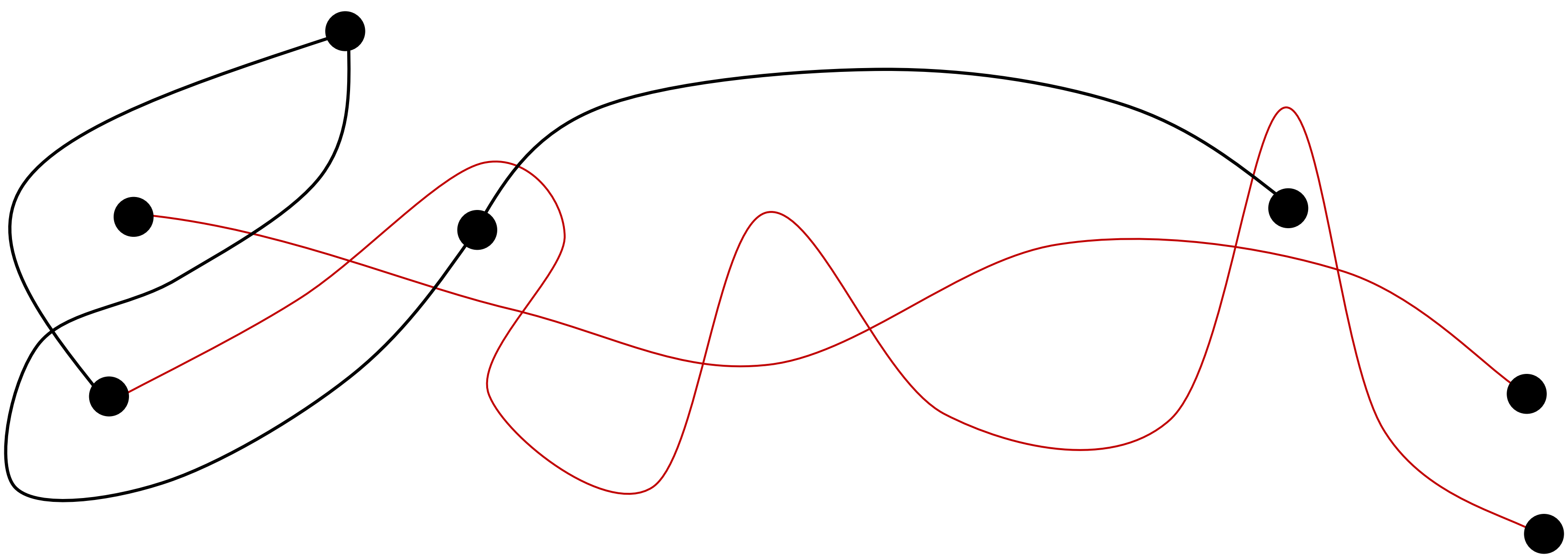


Hanani-Tutte Theorems

Classical Hanani-Tutte

Theorem (Hanani, 1934; Tutte, 1970)

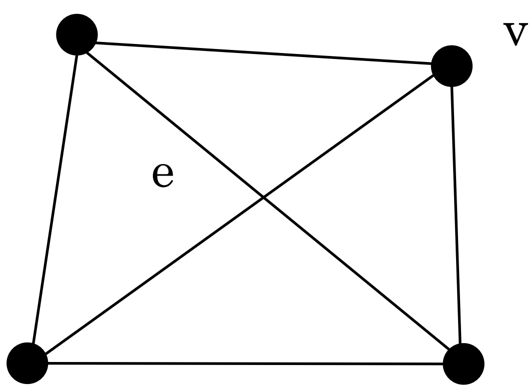
G is planar if and only if G can be drawn so that every two independent edges cross evenly.



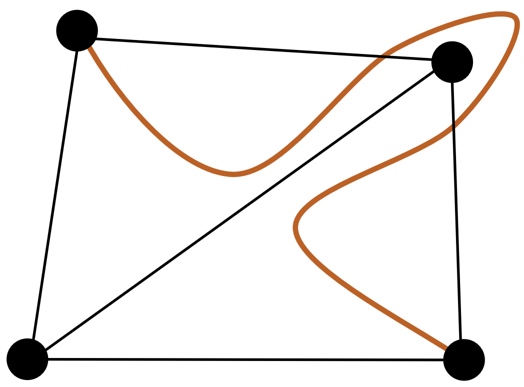
Hanani-Tutte Algebraically

Theorem (Hanani, 1934; Tutte, 1970)

G is planar if and only if G can be drawn so that every two independent edges cross evenly.

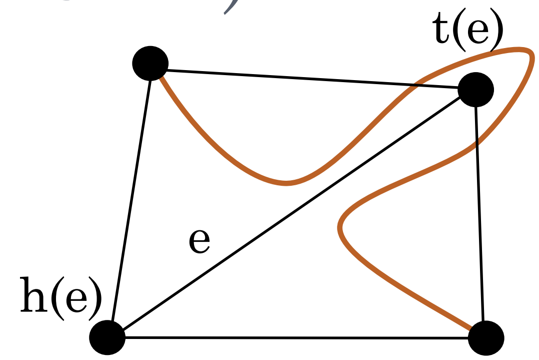


(e,v)-move



→ Yields algebraic system over GF(2)

Algebraic Criterion (Wu, Tutte)



G planar ↔ there is a plane drawing D of G

↔ given any drawing D of G
there is a set of (e,v)-moves so that
in the resulting drawing D'
$i_{D'}(e,f) = 0\ (2)$ for all pairs (e,f) of independent edges in G

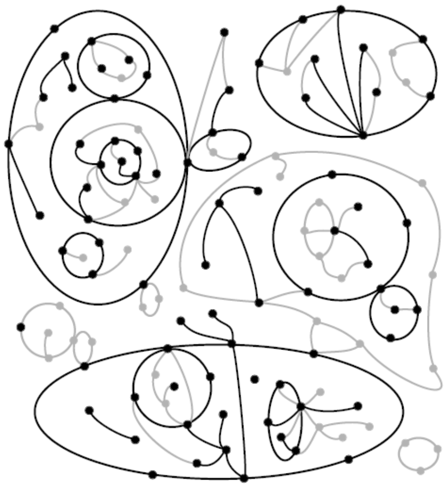
↔ given any drawing D of G
there are x(e,v) in {0,1}, e in E, v in V, so that

$$i_D(e,f) + x(e,t(f)) + x(e,h(f)) + x(f,h(e)) + x(f,t(e)) = 0\ (2)$$

for all pairs (e,f) of independent edges in G

Polynomial time planarity algorithm, $O(n^6)$

Hanani-Tutte for PEP



G: graph
H: subgraph of G
$\mathcal{H}$: embedding of H
(G, H, $\mathcal{H}$): partially embedded graph (PEG)

$\mathcal{H}$

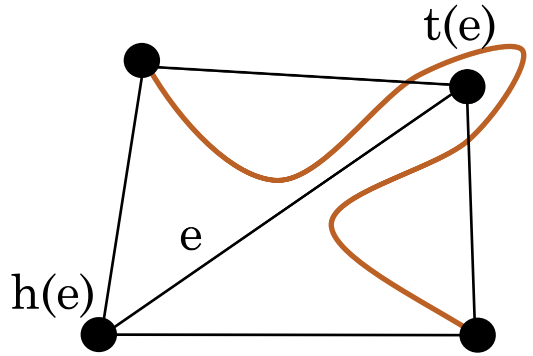
Theorem

PEG (G, H, $\mathcal{H}$) is planar iff G has a drawing

- extending $\mathcal{H}$
- every two independent edges cross evenly.

→ Yields algebraic system over GF(2)

Algebraic Criterion for PEP



(G, H, $\mathcal{H}$) is planar

↔ given any drawing D of G extending $\mathcal{H}$
there are x(e,v) in {0,1}, e in E(G)-E(H), v in V(G) so that

$$i_D(e,f) + x(e,t(f)) + x(e,h(f)) + x(f,h(e)) + x(f,t(e)) = 0 \quad (2)$$

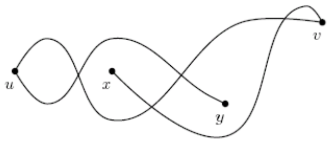
for all pairs (e,f) of independent edges in G

Polynomial time PEP algorithm, $O(n^6)$

A General Hanani-Tutte "Theorem" ?

"G is X-planar iff G has a drawing satisfying X modulo even number of crossings."

HANANI-TUTTE THEOREMS KNOWN FOR



- partially embedded planarity
- partial rotation (with or without flips)
- x-monotonicity (Fulek, Pelsmajer, Schaefer, Stefankovic, 2011)
- level-planarity (implicit in FPSS, 2011)
- c-planarity for 2 clusters and c-connected clustered graphs
  (Fulek, Kyncl, Malinovic, Pālvölgyi, 2013)

→ algebraic systems over GF(2)

→ polynomial time algorithms

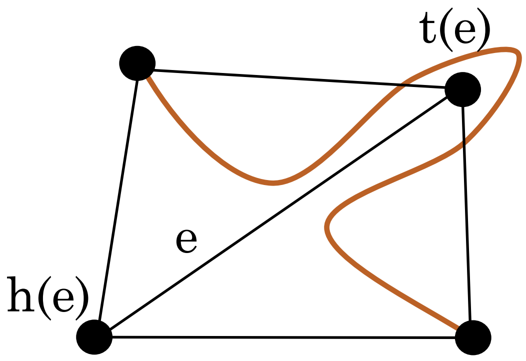
Algorithms for Planarities

- partially embedded planarity
  - linear time, Angelini, Di Battista, Frati, Jelínek, Kratochvíl, Patrignani and Rutter, 2010
  - $O(n^6)$ via HT

- ec-planarity
  - linear time, Gutwenger, Klein, Mutzel, 2008
- partially constrained PQ-planarity
  - linear time for 2-connected graphs, Bläsius, Rutter, 2013
- partial rotation (with or without flips)
  - $O(n^6)$ via HT, linear via SPQR?

- x-monotone, level-planarity
  - linear time, Jünger, Leipert, Mutzel, 1998
  - quadratic time via Hanani-Tutte (FPSS, 2012)

SIOCR CONJECTURE



(G, H) is simultaneously planar

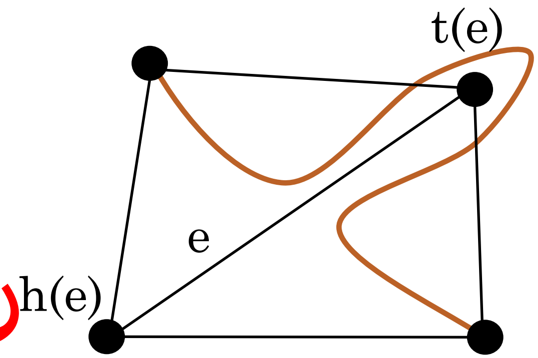
↔ given any drawing D of G u H
there are x(e,v) in {0,1}, e in E(G u H), v in V(G u H) so that

$$i_D(e,f) + x(e,t(f)) + x(e,h(f)) + x(f,h(e)) + x(f,t(e)) = 0 \ (2)$$

for all pairs (e,f) of independent edges so that
e, f in E(G) or e, f in E(H)

→ Yields algebraic systems over GF(2) for simultaneous planarity
→ Yields polynomial time algorithm for simultaneous planarity.
→ Yields polynomial time algorithm for clustered planarity, …

SIOCR CONJECTURE



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for all pairs (e,f) of independent edges so that
e, f in E(G) or e, f in E(H)

Incorrect

→ Yields algebraic systems over GF(2) for simultaneous planarity
→ Yields polynomial time algorithm for simultaneous planarity.
→ Yields polynomial time algorithm for clustered planarity, …

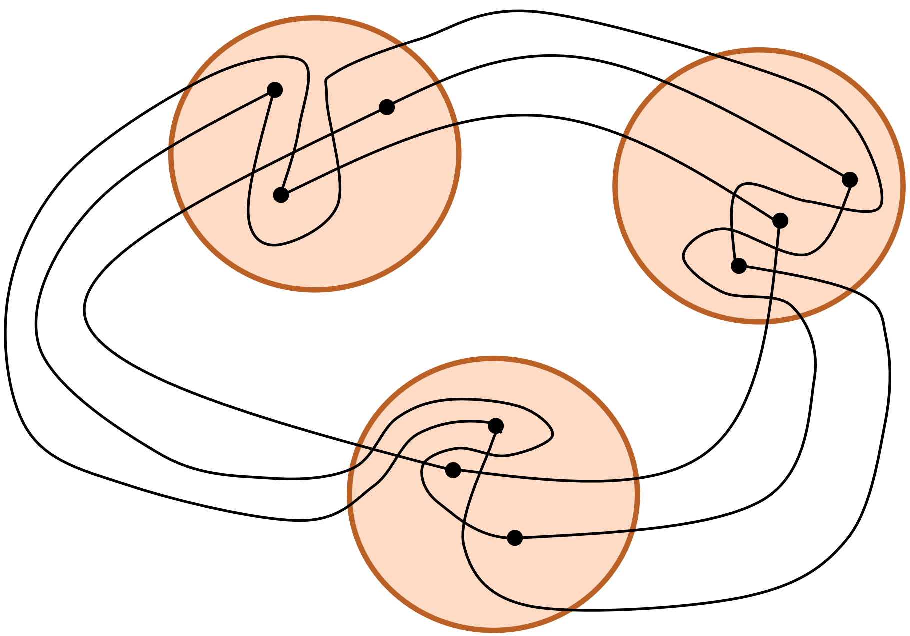
Counterexample

Efficient c-planarity testing algebraically

Radoslav Fulek* Jan Kynčl† Igor Malinović‡ Dömötör Pálvölgyi§

Abstract

We generalize the strong Hanani-Tutte theorem to flat clustered graphs with two disjoint clusters and to clustered graphs with connected clusters, thereby obtaining a new polynomial time algorithm for testing c-planarity in these cases. We also give a new and short proof for a result by Di Battista and Frati about efficient c-planarity testing of an embedded flat clustered graph with small faces based on the matroid intersection algorithm.



But

Theorem

Siocr conjecture true for (G_1,G_2) if:

- $G_1 \cap G_2$ consists of disjoint 2-connected or subcubic components
- one of G_1 or G_2 is disjoint union of subdivisions of 3-connected graph

→ algebraic systems over GF(2)

→ polynomial time algorithms

Algorithms for Simultaneous Planarity

$G_1 \cap G_2$ consists of disjoint cycles
- linear time, Bläsius, Rutter, 2012

$G_1 \cap G_2$ is 2-connected
- linear time, Haeupler, Jampani, Lubiw, 2010

$G_1 \cap G_2$ consists of disjoint 2-connected or subcubic components
- $O(n^6)$ via HT

$G_1 \cap G_2$ is connected, G_1 and G_2 are 2-connected
- quadratic time, Bläsius, Rutter, 2011

one of G_1 or G_2 is disjoint union of subdivision of 3-connected graph
- $O(n^6)$ via HT

An/No Algorithm For Simultaneous Planarity ?

ALGORITHM

given any drawing D of G u H

there are x(e,v) in {0,1}, e in E(G u H), v in V(G u H) so that

$$i_D(e,f) + x(e,t(f)) + x(e,h(f)) + x(f,h(e)) + x(f,t(e)) = 0 \quad (2)$$

for all pairs (e,f) of independent edges so that
e, f in E(G) or e, f in E(H)

G = G u H, n = |V(GuH)|, m = |E(GuH)|

- (e,v)-moves: n*m
- (e,f)-variables: m*m

So (m*m) * (n*m) system

- Gaussian elimination in cubic time to solve, so $O(n^6)$

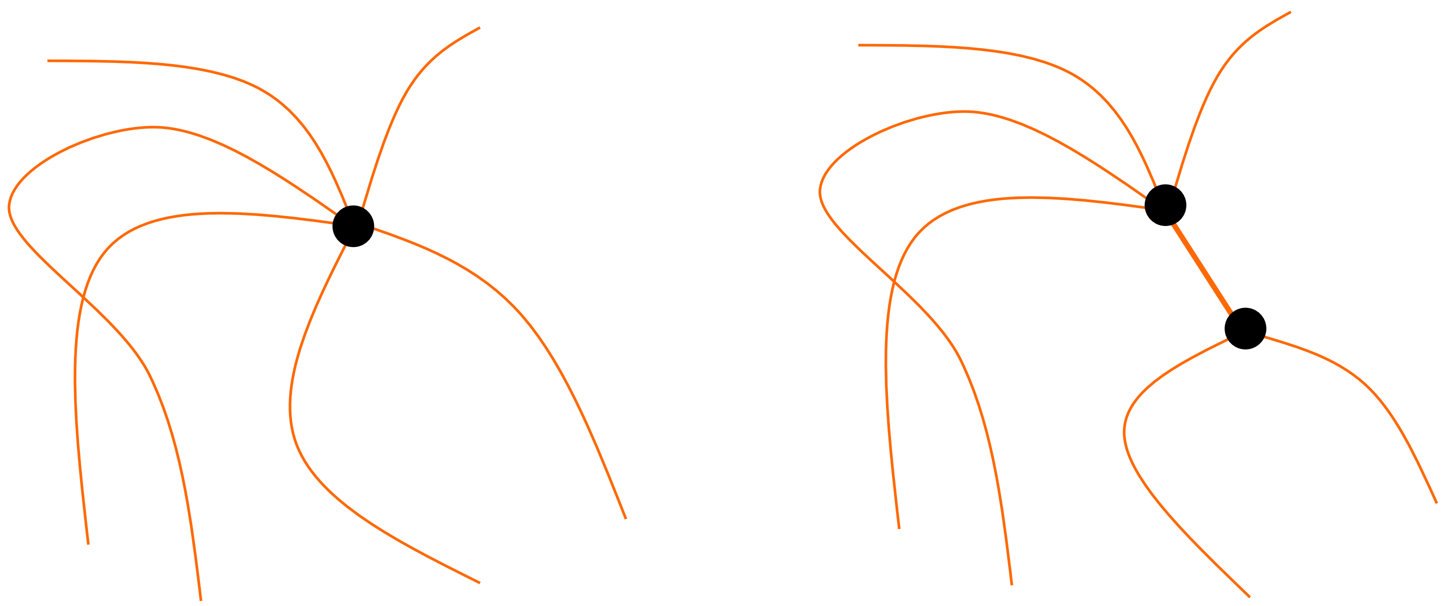
Sparse Solver in Python

	V	E	ef	ev	Pre-proces	Solve	Total
G4	14	20	102	216	0.01s	0.007 s	0.017s
G6	29	63	1213	1654	0.2s	0.5s	0.8s
G7	50	118	4489	5574	2.5s	5.8s	8.3s
G12	50	175	9376	8400	7.8s	34s	42s
G13	100	251	21405	24598	63s	581s	644s
G11	250	352	47413	78626	707s	2920s	3627s

Is the Algorithm Correct?

Yes, no, and don't know, but we can check it:

If it says yes: use self-reducibility to verify truth
Base case: subcubic graphs



In M4RI processing should take <1s, verification 1 minute
Verification also gives embedding

Apropos Correctness

With verification, the algorithm gives three answers:

Yes, the graph is simultaneously planar

True, because of verification

No, the graph is not simultaneously planar

True: scr $(G_1, G_2) = 0$ implies siocr $(G_1, G_2) = 0$

Input is counterexample to conjecture

True: there will be a counterexample.

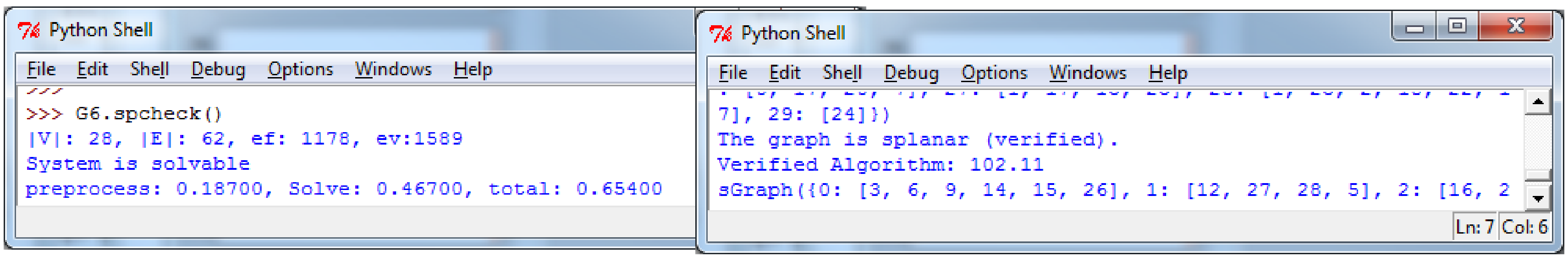
+ list of cases for which it *is* correct

- $G_1 \cap G_2$ consists of disjoint 2-connected or subcubic components
- one of G_1 or G_2 is disjoint union of subdivision of 3-connected graph

Verified Algorithm

G6: 30 vertices, 63 edges:

G6 = {0: [3, 6, 26, 9, 14, 15], 1: [12, 28, 5, 27], 2: [16, 13, 5, 28], 3: [4, 16, 17, 0, 6, 5], 4: [10, 3], 5: [6, 2, 1, 3, 12, 10, 21], 6: [5, 23, 25, 0, 3, 16], 7: [20, 13, 12, 26, 19], 8: [20], 9: [0, 17], 10: [15, 4, 28, 5, 13, 20], 11: [], 12: [1, 5, 7], 13: [2, 7, 21, 10, 16, 24], 14: [15, 17, 0, 16], 15: [10, 27, 14, 0], 16: [2, 3, 25, 19, 13, 14, 6], 17: [28, 27, 3, 26, 9, 14, 22], 18: [28, 20], 19: [16, 7], 20: [7, 8, 26, 18, 10], 21: [25, 5, 13, 28], 22: [26, 28, 17], 23: [27, 29, 6], 24: [29, 13], 25: [21, 16, 27, 28, 6], 26: [7, 0, 17, 20, 22], 27: [23, 17, 25, 1, 15], 28: [17, 1, 10, 25, 18, 2, 22, 21], 29: [23, 24]}



Verification is $O(n^8)$; in this case took 102 seconds

Verification also gives embedding

Verified Algorithm in Python

	V	E	ef	ev	Total
G4	14	20	102	216	0.5s
G6	29	63	1213	1654	103s
G7	50	118	4489	5574	2365s
G12	50	175	9376	8400	25756s
G13	100	251	21405	24598	?
G11	250	352	47413	78626	?

Algorithm: To Do

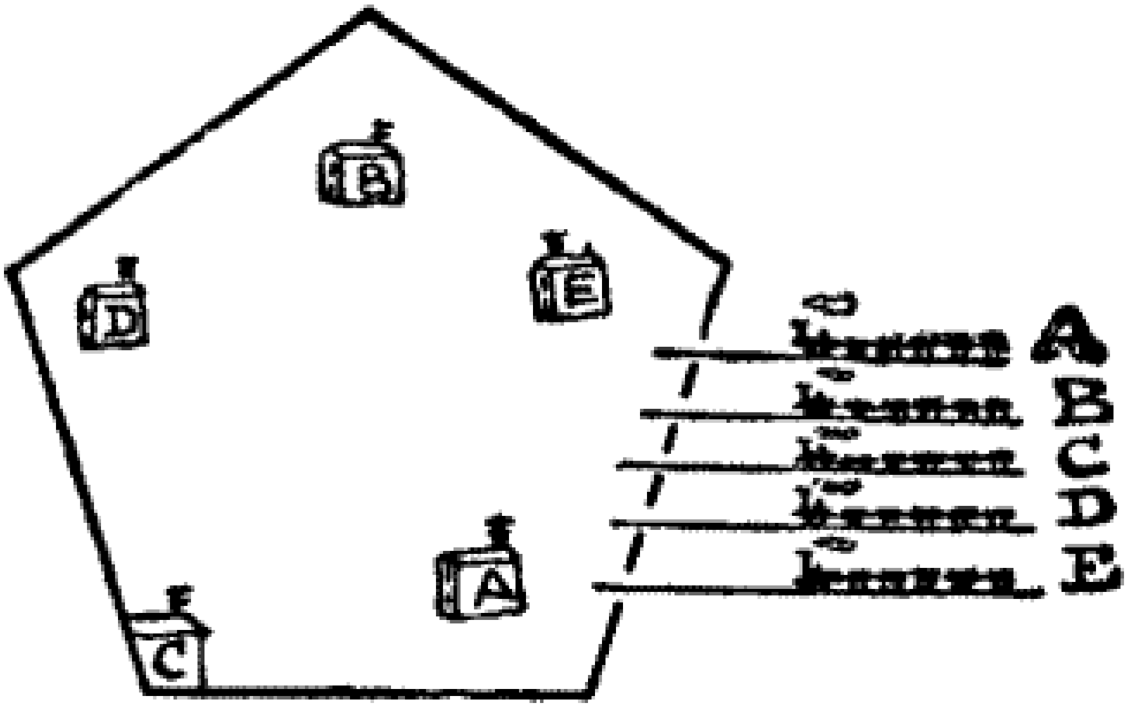
- Modify so it finds an embedding (similar to verification)
  - can be done: upcoming work with Mutzel, Gutwenger

- Improve algorithm?
  - Planarity can be improved to O(n)
  - Level-planarity can be improved to $O(n^2)$

  - PEP? SP? For special cases?
  - Use structure of $G_1 \cap G_2$ to speed up verified algorithm
  - c-planarity: C++ implementation, upcoming work with Mutzel, Gutwenger

- Save siocr conjecture by modifying it?

Chinese Railways (Dudeney)



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