Generalized Inversion Sequences

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Permutations and Descents

S_n: set of permutations $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$

$\mathrm{Des}\,\pi$: $\{i \in \{1, \ldots n-1\} \mid \pi(i) > \pi(i+1)\}$ (*descents*)

$\mathrm{des}\,\pi$: $|\mathrm{Des}\,\pi|$, the number of descents.

$\pi \in S_3$	$\mathrm{Des}\,\pi$	$\mathrm{des}\,\pi$
1 2 3	$\{\ \}$	0
1 3 2	$\{2\}$	1
2 1 3	$\{1\}$	1
2 3 1	$\{2\}$	1
3 1 2	$\{1\}$	1
3 2 1	$\{1, 2\}$	2

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Descent polynomial:

$$E_n(x) = \sum_{\pi \in S_n} x^{\mathrm{des}\,\pi}$$

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Eulerian polynomials: $E_n(x)$

The Eulerian polynomials, $E_n(x)$

$$E_n(x) = \sum_{\pi \in S_n} x^{\mathrm{des}\,\pi}$$

$$\sum_{t \geq 0} (t+1)^n x^t = \frac{E_n(x)}{(1-x)^{n+1}}$$

$$\sum_{n \geq 0} E_n(x) \frac{z^n}{n!} = \frac{(1-x)}{e^{z(x-1)} - x}$$

Inversion Sequences

$$\mathbf{I}_n = \{(e_1, \ldots e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < i\}$$

Encode permutations as inversion sequences $\phi : S_n \to \mathbf{I}_n$
$\phi(\pi) = (e_1, \ldots, e_n)$, where

$$e_j = |\{i \mid i < j \text{ and } \pi(i) > \pi(j)\}|.$$

Example:

$$\phi(4\ 3\ 6\ 5\ 1\ 2) = (0, 1, 0, 1, 4, 4).$$

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$$\phi(4\ 3\ 6\ 5\ 1\ 2) = (0, 1, 0, 1, 4, 4).$$

Claim: ϕ is a bijection with $\mathrm{Des}\,\pi = \mathrm{Asc}\,\phi(\pi)$. What is "$\mathrm{Asc}$"?

Ascents

What is an “ascent” in an inversion sequence?

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Lemma

If $0 \le e_j < j$ *for all* $j \le n$, *then for* $1 \le i < n$,

$$e_i < e_{i+1} \quad \text{iff} \quad \frac{e_i}{i} < \frac{e_{i+1}}{i+1}.$$

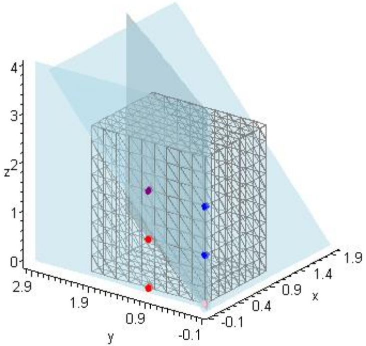
View inversion sequences as lattice points

in a (half-open) $1 \times 2 \times \cdots \times n$ box

View ascent constraints as hyperplane constraints:

$$0 < e_1 \text{ and } \frac{e_i}{i} < \frac{e_{i+1}}{i+1}, \quad 1 \leq i < n$$

$\pi \in S_3$	$e \in \mathbf{I}_n$	Asc e
1 2 3	(0,0,0)	{ }
1 3 2	(0,0,1)	{2}
2 1 3	(0,1,0)	{1}
2 3 1	(0,0,2)	{2}
3 1 2	(0,1,1)	{1}
3 2 1	(0,1,2)	{1, 2}

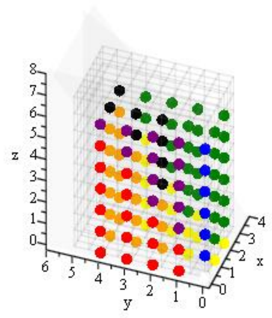


s-inversion sequences

For any sequence $s = (s_1, s_2, \ldots, s_n)$ of positive integers:

$$\mathbf{I}_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

(lattice points in a half-open $s_1 \times s_2 \times \cdots \times s_n$ box)



$$\left|\mathbf{I}_n^{(s)}\right| = s_1 s_2 \cdots s_n$$

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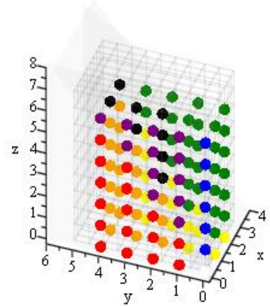

An *ascent* of e is a position i:
$1 \le i < n$ and

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}.$$

If $e_1 > 0$ then 0 *is an ascent*.

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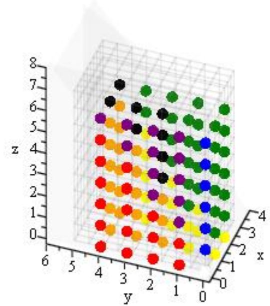


Example: $(2, 4, 5) \in \mathbf{I}_n^{(3,5,7)}$

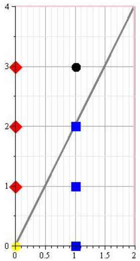
$$\mathrm{Asc}\, e = \{0, 1\}$$

$2 \notin \mathrm{Asc}\, e$ since $4/5 \not< 5/7$

Ascent polynomials of s-inversion sequences

$$A_n^{(s)}(x) = \sum_{e \in \mathbf{I}_n^{(s)}} x^{\operatorname{asc} e}$$

$(2, 4)$-inversion sequences



Ascent sets:
$\{\ \}$ yellow dot
$\{0\}$ blue square
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$$A_n^{(2,4)}(x) = 1 + 6x + x^2$$

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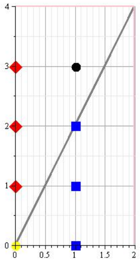

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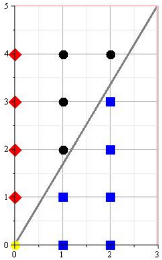
$(3,5)$-inversion sequences



$A_n^{(3,5)}(x) = 1 + 10x + 4x^2$

Call $A_n^{(s)}(x)$ the *s-Eulerian polynomial* since,

when $s = (1, 2, \ldots, n)$,

$$A_n^{(s)}(x) = \sum_{e \in \mathbf{I}_n^{(s)}} x^{\operatorname{asc} e} = \sum_{\pi \in S_n} x^{\operatorname{des} \pi} = E_n(x),$$

the Eulerian polynomial.

(Recall bijection $\phi : S_n \to \mathbf{I}_n$ with $\operatorname{Des} \pi = \operatorname{Asc} \phi(\pi)$)

Why s-inversion sequences?

- Natural model for combinatorial structures
- Can prove general properties of the s-Eulerian polynomials
- Surprising results follow using Ehrhart theory
- Can be encoded as lecture hall partitions
- Lead to a natural refinement of the s-Eulerian polynomials
- Help answer questions about lecture hall partitions

sequence s	s-Eulerian polynomial
$(1,2,3,4,5,6)$	$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$
$(2,4,6,8,10)$	$1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5$
$(6,5,4,3,2,1)$	$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$
$(1,1,3,2,5,3)$	$1 + 20x + 48x^2 + 20x^3 + x^4$
$(1,3,5,7,9,11)$	$1 + 358x + 3580x^2 + 5168x^3 + 1328x^4 + 32x^5$.
$(7,2,3,5,4,6)$	$1 + 71x + 948x^2 + 2450x^3 + 1411x^4 + 159x^5$

1.

Signed permutations
and
$(2, 4, 6, \ldots, 2n)$-inversion sequences

$$\left| \mathbf{I}_n^{(2,4,6,\ldots,2n)} \right| = 2^n n!$$

Signed Permutations B_n

$$B_n = \{(\sigma_1, \ldots, \sigma_n) \mid \exists \pi \in S_n,\ \forall i\ \sigma_i = \pm\pi(i)\}$$

$\pi \in B_n$	$\mathrm{Des}\,\pi$
(1, 2)	$\{\ \}$
(-1, 2)	$\{0\}$
(1,-2)	$\{1\}$
(-1,-2)	$\{0,1\}$
(2, 1)	$\{1\}$
(-2, 1)	$\{0\}$
(2,-1)	$\{1\}$
(-2,-1)	$\{0\}$

$$\mathrm{Des}\,\sigma = \{i \in \{0, \ldots, n-1\} \mid \sigma_i > \sigma_{i+1}\},$$

with the convention that $\sigma_0 = 0$.

descent polynomial:

$$1 + 6x + x^2$$

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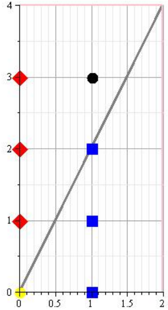
$(2,4)$-inversion sequences



descent polynomial:

$$1 + 6x + x^2$$

ascent polynomial:

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Theorem (Pensyl, S 2012/13)

$$\sum_{\sigma \in B_n} x^{\operatorname{des} \sigma} = A_n^{(2,4,\ldots,2n)}(x).$$

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Proof.

There is a bijection $\Theta : B_n \to \mathbf{I}_n^{(2,4,\ldots,2n)}$ satisfying

$$\mathrm{Des}\,\sigma = \mathrm{Asc}\,\Theta(\sigma).$$

□

2.

$(s_1, s_2, \ldots, s_n)$-inversion sequences

vs.

$(s_n, s_{n-1}, \ldots, s_1)$-inversion sequences

Example from table:

$$A_6^{(1,2,3,4,5,6)}(x) = 1 + 57\,x + 302\,x^2 + 302\,x^3 + 57\,x^4 + x^5$$

$$= A_6^{(6,5,4,3,2,1)}(x)$$

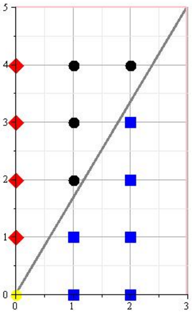
Theorem (S, Schuster 2012; Liu, Stanley 2012)

For any sequence $(s_1, s_2, \ldots, s_n)$ *of positive integers,*

$$A_n^{(s_1,s_2,\ldots,s_n)}(x) \;=\; A_n^{(s_n,s_{n-1},\ldots,s_1)}(x).$$

Reversing s preserves the ascent polynomial

$(3,5)$-inversion sequences



$1 + 10\,x + 4\,x^2$

Ascent sets:

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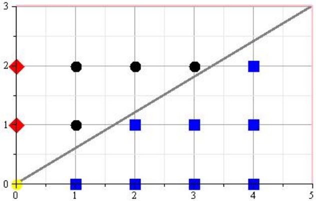
$(5,3)$-inversion sequences



$1 + 10\,x + 4\,x^2$

but *not* necessarily the partition into ascent sets

Roots of s-Eulerian polynomials

Example from table:

$$A_6^{(7,2,3,5,4,6)}(x) \;=\; 1 + 71\,x + 948\,x^2 + 2450\,x^3 + 1411\,x^4 + 159\,x^5$$

Roots in the intervals:

$$[[-19/1024, -9/512], [-77/1024, -19/256],$$

$$[-423/1024, -211/512], [-1701/1024, -425/256],$$

$$[-3435/512, -6869/1024]]$$

Theorem (S, Visontai 2012)

For every sequence s of positive integers, $A_n^{(s)}(x)$ *has all real roots.*

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Corollary (Frobenius 1910; Brenti 1994)

The descent polynomials for Coxeter groups of types A and B have all real roots.

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New: ([S, Visontai 2013]) Method can be adapted to type D.

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For every sequence s of positive integers, $A_n^{(s)}(x)$ has all real roots.

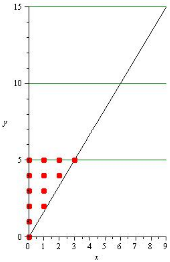
Corollary

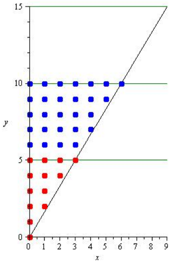
For any s, the sequence of coefficents of the s-Eulerian polynomial is unimodal and log-concave.

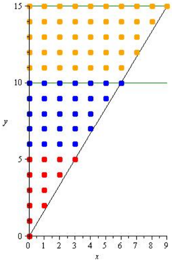
Example $A_6^{(7,2,3,5,4,6)}(x)$:

$$1, \ 71, \ 948, \ 2450, \ 1411, \ 159, \ 1$$

Lecture Hall Polytopes
and
Ehrhart Theory



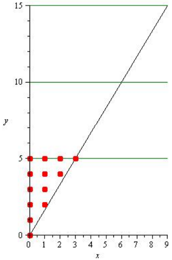




Lecture hall polytopes

s-lecture hall polytope:

$$\mathbf{P}_n^{(s)} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$



$\mathbf{P}_2^{(3,5)}$

Lecture hall polytopes

s*-lecture hall polytope*:

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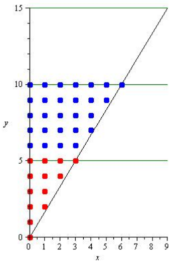

$\mathbf{P}_2^{(3,5)}$

t-th *dilation* of $\mathbf{P}_n^{(\mathbf{s})}$:

$$t\mathbf{P}_n^{(s)} = \{ t\lambda \mid \lambda \in \mathbf{P}_n^{(s)} \}$$

Lecture hall polytopes

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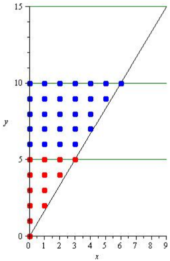

$\mathbf{P}_2^{(3,5)}$

t-th dilation of $\mathbf{P}_n^{(s)}$:

$$t\mathbf{P}_n^{(s)} = \{ t\lambda \mid \lambda \in \mathbf{P}_n^{(s)} \}$$

Ehrhart polynomial of $\mathbf{P}_n^{(s)}$:

$$i(\mathbf{P}_n^{(s)}, t) = |t\mathbf{P}_n^{(s)} \cap \mathbb{Z}^n|.$$

Lecture hall polytopes

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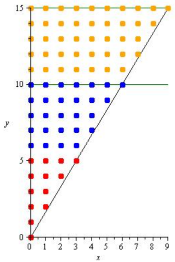

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Connection between lecture hall polytopes and inversion sequences

Theorem (S, Schuster 2012)

For any sequence s *of positive integers,*

$$\sum_{t \geq 0} i(\mathbf{P}_n^{(s)}, t)\, x^t \quad = \quad \frac{\sum_{e \in \mathbf{I}_n^{(s)}} x^{\mathrm{asc}\,(e)}}{(1-x)^{n+1}}.$$

5.

The sequences

$$s = (1,\ 1,\ 3,\ 2,\ 5,\ 3,\ 7,\ 4, \ldots)$$

and

$$s = (1,\ 4,\ 3,\ 8,\ 5,\ 12,\ 7,\ 16, \ldots)$$

Note:

$$\left| \mathbf{I}_{2n}^{(1,1,3,2,5,3,\ldots,2n-1,n)} \right| = n!(1 \cdot 3 \cdot 5 \cdot \cdots \cdot 2n-1) = \frac{(2n)!}{2^n}$$

Theorem (S, Visontai 2012)

$A_{2n}^{(1,1,3,2,\ldots,2n-1,n)}$ is the descent polynomial for permutations of the multiset $\{1,1,2,2,\ldots,n,n\}$.

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Bijective proof?

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Bijective proof?

Conjecture (S, Visontai 2012)

$A_{2n}^{(1,4,3,8,\ldots,2n-1,4n)}$ is the descent polynomial for the signed permutations of $\{1, 1, 2, 2, \ldots, n, n\}$.

6.

The sequences $s = (1, k+1, 2k+1, 3k+1, \ldots)$

$$k = 1: \quad (1,\ 2,\ 3,\ 4,\ 5,\ \ldots)$$
$$k = 2: \quad (1,\ 3,\ 5,\ 7,\ 9,\ \ldots)$$

6.

The sequences $s = (1, k+1, 2k+1, 3k+1, \ldots)$

$$\begin{aligned} k = 1: &\quad (1,\ 2,\ 3,\ 4,\ 5,\ \ldots) \\ k = 2: &\quad (1,\ 3,\ 5,\ 7,\ 9,\ \ldots) \end{aligned}$$

Let:

$$\begin{aligned} \mathbf{I}_{n,k} &= \mathbf{I}_n^{(1,k+1,2k+1,\ldots,(n-1)k+1)} \\ A_{n,k}(x) &= A_n^{(1,k+1,2k+1,\ldots,(n-1)k+1)}(x) \end{aligned}$$

Recall $A_{n,1}(x) = E_n(x)$.

The $1/k$-Eulerian polynomials

Theorem (S, Viswanathan 2012)

For positive integer k*,*

$$
\begin{aligned}
A_{n,k}(x) &= \sum_{e \in \mathbf{I}_{n,k}} x^{\operatorname{asc} e} \\
\sum_{t \geq 0} \binom{t-1+\frac{1}{k}}{t} (kt+1)^n x^t &= \frac{A_{n,k}(x)}{(1-x)^{n+\frac{1}{k}}} \\
\sum_{n \geq 0} A_{n,k}(x) \frac{z^n}{n!} &= \left(\frac{1-x}{e^{kz(x-1)} - x} \right)^{\frac{1}{k}}
\end{aligned}
$$

Theorem (S, Viswanathan 2012)

$$\sum_{e \in \mathbf{I}_{n,k}} x^{\operatorname{asc} e} = \sum_{\pi \in S_n} x^{\operatorname{exc} \pi} k^{n - \#\operatorname{cyc} \pi},$$

where

$$\operatorname{exc} \pi = |\{i \mid \pi(i) > i\}|$$

and $\#\operatorname{cyc} \pi$ *is the number of cycles in the disjoint cycle representation of* π.

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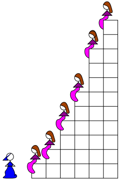
Combinatorial proof?

7.

Lecture Hall Partitions

$$L_n = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n} \right\}$$



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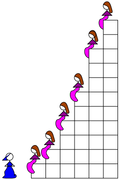

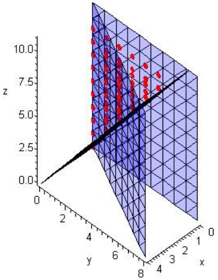

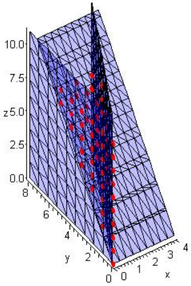

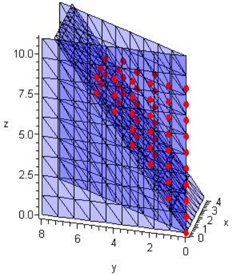

s-lecture hall partitions

$$L_n^{(s)} = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}$$

Theorem (Bousquet-Mélou, Eriksson 1997)

For $s = (1, 2, \ldots n)$,

$$\sum_{\lambda \in L_n^{(s)}} q^{|\lambda|} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})},$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

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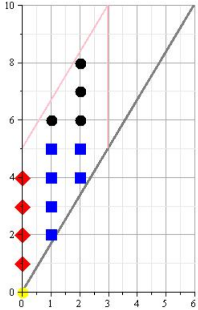
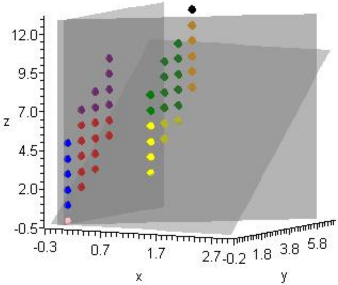
(What other sequences s *give rise to nice generating functions?)*

8.

Fundamental Lecture Hall Parallelepiped



$s = (3, 5)$



$s = (2, 4, 6)$

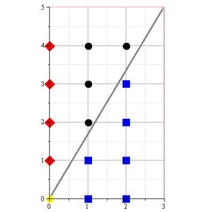
Fundamental (half-open) s-lecture hall parallelepiped:

$$\Pi_n^{(s)} = \left\{ \sum_{i=1}^{n} c_i w_i \mid 0 \leq c_i < 1 \right\},$$

where $w_i = [0, \ldots, 0, s_i, s_{i+1}, \ldots s_n]$.

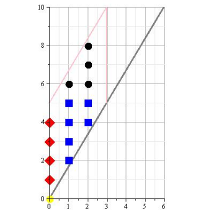
Theorem (Liu, Stanley 2012)

There is a bijection between $I_n^{(s)}$ and $\Pi_n^{(s)} \cap \mathbb{Z}^n$.

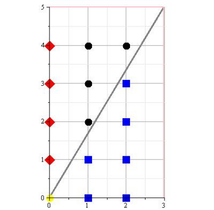


$I_2^{(3,5)}$

$\longrightarrow$



$\Pi_2^{(3,5)} \cap \mathbb{Z}^2$



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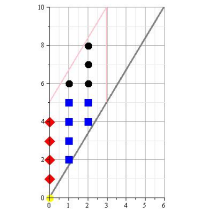

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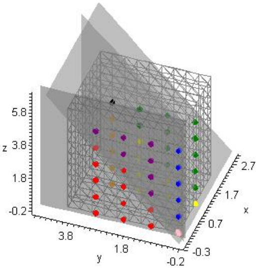
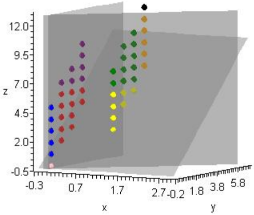


$\longrightarrow$



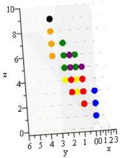
$\mathbf{I}_3^{(2,4,6)}$

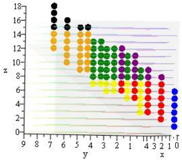
$\Pi_3^{(2,4,6)} \cap \mathbb{Z}^3$

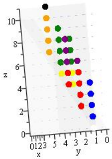
9.

Inflated s-Eulerian polyomials

$$\sum_{\lambda \in \Pi_n^{(s)}} x^{\lambda_n}$$







Define the *inflated s-Eulerian polyomial* by

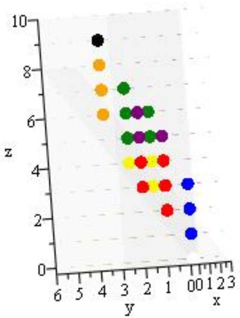
$$Q_n^{(s)}(x) = \sum_{\lambda \in \Pi_n^{(s)}} x^{\lambda_n}.$$

Theorem (Pensyl,S 2013)

For any sequence s of positive integers,

$$Q_n^{(s)}(x) \;=\; \sum_{e \in \mathbf{I}_n^{(s)}} x^{s_n \mathrm{asc}\, e - e_n}$$

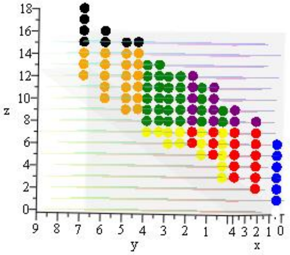
For $s = (1, 2, \ldots, n)$, the coefficient sequence of $Q_n^{(s)}$ gives an interesting refinement of the Eulerian numbers:



Coefficient sequence:

1, 1, 2, 4, 4, 4, 4, 2, 1, 1

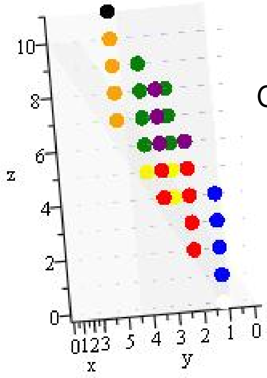
For $s = (1, 3, \ldots, 2n-1)$, the coefficient sequence of $Q_n^{(s)}(x)$ is symmetric (but not the coefficient sequence of $A_n^{(s)}(x)$.)



Coefficient sequence:

1, 1, 2, 4, 4, 6, 9, 10, 10, 11, 10, 10, 9, 6, 4, 4, 2, 1, 1

For $s = (1, 1, 2, 3, 5, 8, \ldots)$, the coefficient sequence of $Q_n^{(s)}$ is not symmetric for $n \geq 5$.



Coefficient sequence:

1, 1, 2, 2, 4, 4, 4, 4, 4, 2, 1, 1,

Gorenstein cones and self-reciprocal generating functions

Self-reciprocal:

satisfies $f(q) = q^b f(1/q)$ for some nonnegative integer b

Examples:

$$1 + x + 2x^2 + 4x^3 + 4x^4 + 4x^5 + 4x^6 + 2x^7 + x^8 + x^9$$

$$\frac{1}{(1-q)(1-q^3)(1-q^5)}$$

A pointed rational cone $C \subseteq \mathbb{R}^n$ is *Gorenstein* if there exists a point c in the interior C^0 of C such that $C^0 \cap \mathbb{Z}^n = c + (C \cap \mathbb{Z}^n)$.

Theorem (Special case of a result due to Stanley 1978)

The s-lecture hall cone is Gorenstein if and only if $Q_n^{(s)}(x)$ is self-reciprocal; also, if and only if the following is self reciprocal:

$$f_n^{(s)}(q) = \sum_{\lambda \in L_n^{(s)}} q^{|\lambda|}$$

Theorem (Bousquet-Mélou, Eriksson 1997; Beck, Braun, Köppe, S, Zafeirakopoulos 2012)

The s-lecture hall cone is Gorenstein if and only if there exists $c \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j-1})$$

for $j > 1$ with $c_1 = 1$.

Theorem (BBKSZ)

[Beck, Braun, Köppe, S, Zafeirakopoulos 2012]

Let s *be a sequence of positive integers defined by*

$$s_n = \ell s_{n-1} + m s_{n-2}, (*)$$

with $s_0 = 0, s_1 = 1$*. Then the* s*-lecture hall cone in Gorenstein for all* n *if and only if* $m = -1$*.*

Theorem (BBKSZ)

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Let s be a sequence of positive integers defined by

$$s_n = \ell s_{n-1} + m s_{n-2}, (*)$$

with $s_0 = 0, s_1 = 1$. Then the s-lecture hall cone in Gorenstein for all n if and only if $m = -1$.

Sequences (*) with $m = -1$ are called ℓ-*sequences*.

ℓ-sequences

$$s_n = \ell s_{n-1} - s_{n-2},$$

with $s_0 = 1$, $s_1 = 1$.

$\ell = 2$

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9, \ldots$$

$\ell = 3$

$$1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377,\ 987,\ 2584, \ldots$$

Theorem (Bousquet-Mélou, Eriksson 1997)

If s is an ℓ-sequence,

$$\sum_{\lambda \in L_n^{(s)}} q^{|\lambda|} = \frac{1}{(1-q)(1-q^{s_1+s_2})(1-q^{s_2+s_3})\cdots(1-q^{s_{n-1}+s_n})}.$$

Conversely, by the BBKSZ Theorem, for a sequence of the form (*) unless s is an ℓ-sequence, the s-lecture hall partitions cannot, for all n, have a generating function of the form

$$\frac{1}{(1-q^{c_1})(1-q^{c_2})\cdots(1-q^{c_n})}.$$

Question

What combinatorial family is being represented by the s-inversion sequences when s is an ℓ-sequence?

(When $\ell = 2$, the answer is permutations.)

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Thank you!