Covering with intervals in distributive lattices

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June 13, 2013

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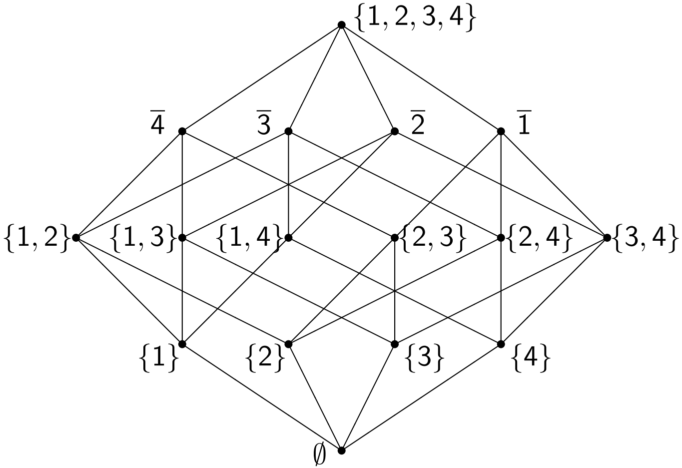
Therefore the family of intervals

$$[1, \overline{2}], [2, \overline{3}], \ldots, [n-1, \overline{n}], [n, \overline{1}]$$

covers $B_n - \{\mathbf{0}, \mathbf{1}\}$. □

Example: $n = 4$.



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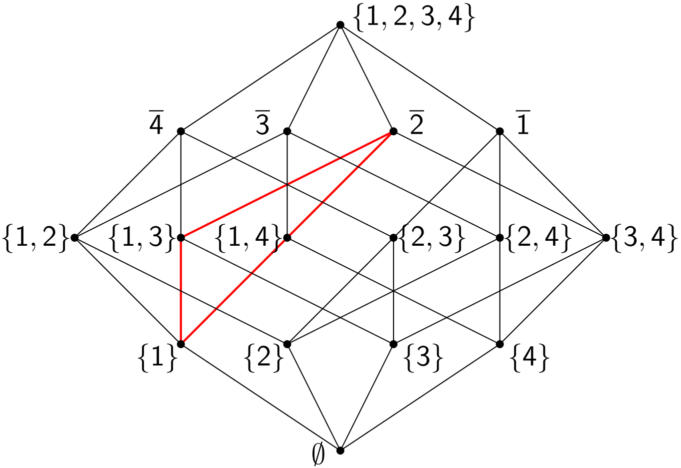

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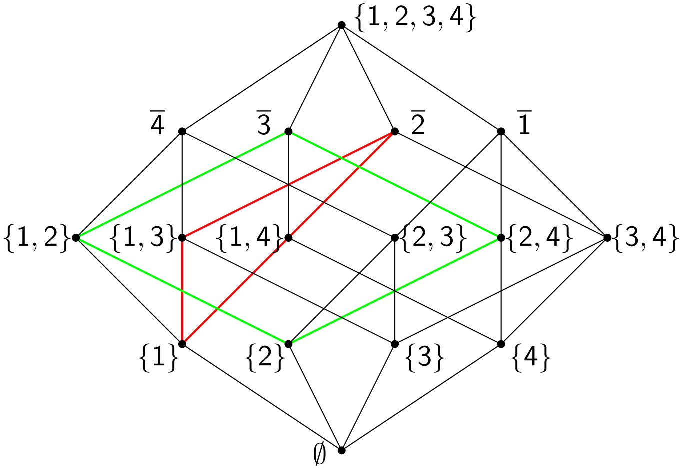

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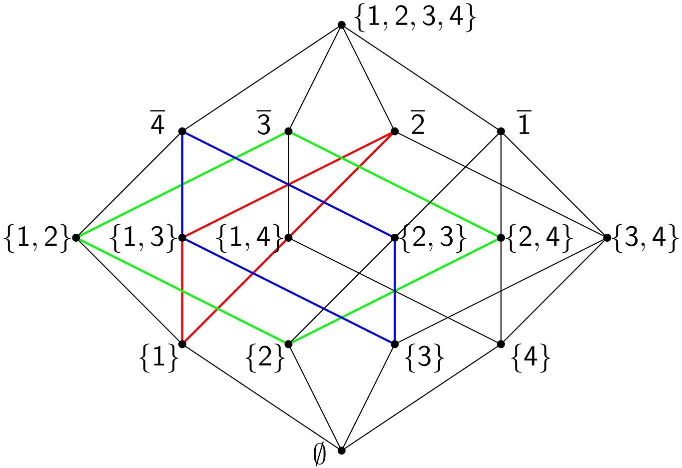

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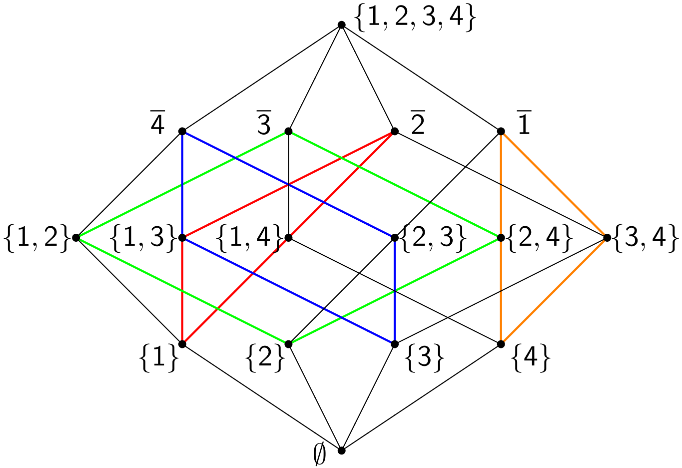

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This result can be written: given the levels L_1 of all atoms and L_{n-1} of all coatoms of B_n, the convex hull

$$[L_1, L_{n-1}] = \{X \in B_n \mid A \subseteq X \subseteq B \text{ for some } A \in L_1, B \in L_{n-1}\}$$

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In fact, this result can be extended to arbitrary levels of a Boolean lattice.

Theorem

(D. Duffus, B. Sands) For any two levels L_k *and* L_ℓ *of a Boolean lattice* B_n*, where* $1 \leq k < \ell < n$*, the convex hull* $[L_k, L_\ell]$ *can be covered by* $\max(\binom{n}{k}, \binom{n}{\ell})$ *intervals* $[X, Y]$*, where* $|X| = k$ *and* $|Y| = \ell$ *for each* X, Y*.*

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Given an element X of B_n (written as a 0–1 sequence of length n), proceed from left to right to match each 1 in X with the nearest unmatched 0 (if any) to its left. Then change the first unmatched 0 in X to 1. This defines the next element (if it exists) in the chain containing X.

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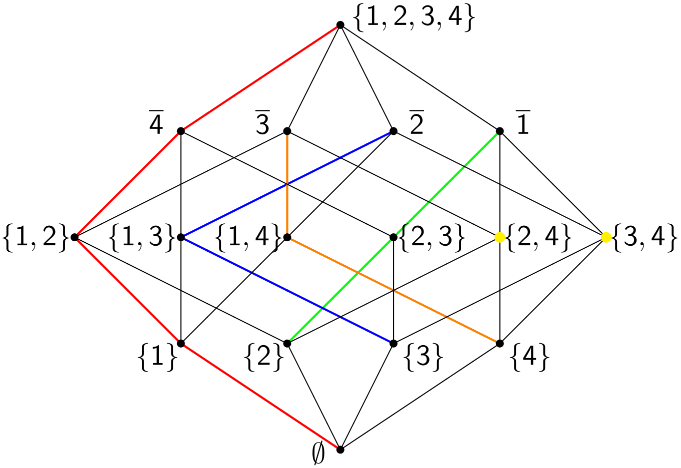
which accounts for all elements of B_4.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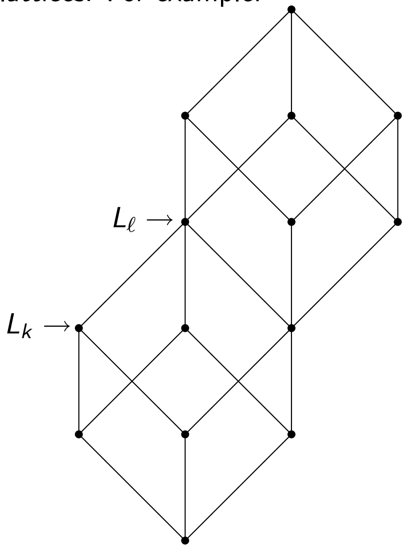

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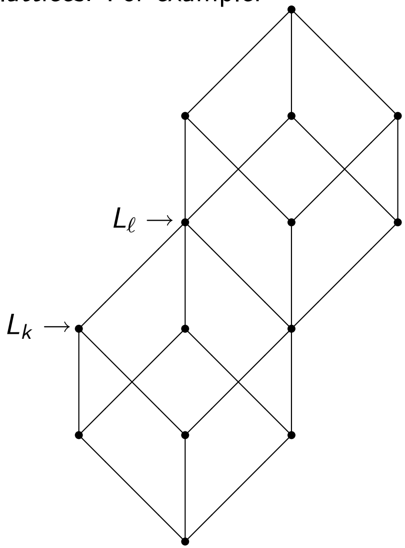

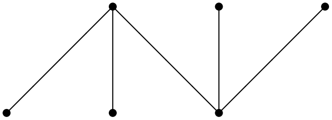

Requires 4 intervals,

while $|L_k| = |L_\ell| = 3$.

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We use $|L_k| + |L_\ell| - 1$ instead of $|L_k| + |L_\ell| - 2$ to account for the trivial case $|L_k| = 1$ or $|L_\ell| = 1$.

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Is this result still true if A is the set of all atoms but C is replaced by an arbitrary level of a distributive lattice L?

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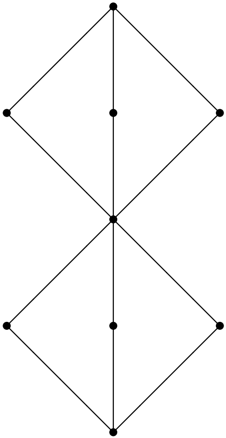

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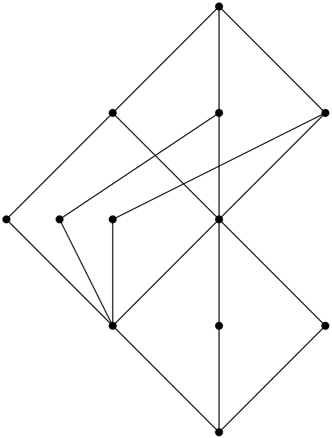

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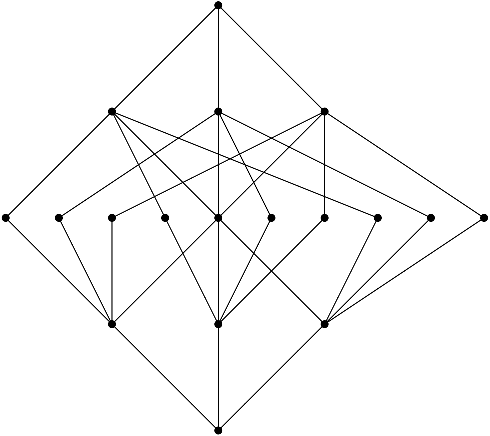

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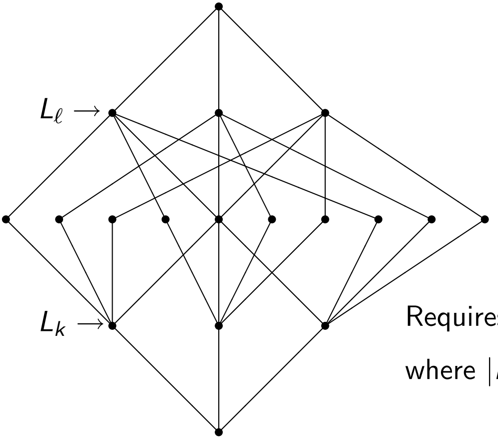

Requires $3 \times 3 = 9$ intervals, where $|L_k| = |L_\ell| = 3$.

Thank you!