Hedetniemi conjecture for strict vector chromatic number

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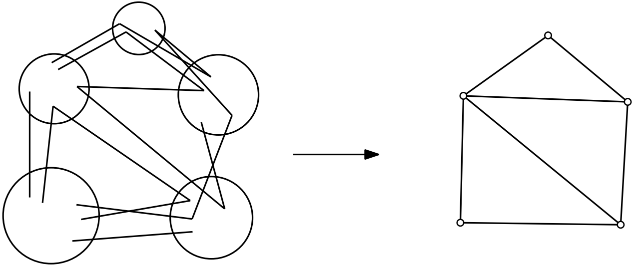
Outline

Graph homomorphism

Graph homomorphism is $\varphi : V(G) \to V(G)$ such that

$$u \sim v \Rightarrow \varphi(u) \sim \varphi(v)$$



Monotone graph parameters

Graph parameter $f : \text{Graphs} \to \mathbb{R}$ is *monotone* if

$$G \to H \Rightarrow f(G) \leq f(H)$$

Examples: $\chi, \chi_c, \chi_f, \ldots$

Graph products

G, H – graphs. Their products have vertex set $V(G) \times V(H)$ and adjacency defined so, that $(g_1, h_1) \sim (g_2, h_2)$ iff

- $g_1 \sim g_2$ and $h_1 \sim h_2$ — *categorical product* $G \times H$
- $g_1 \sim g_2$ and $h_1 = h_2$ OR vice versa — *cartesian product* $G \square H$
- $g_1 \sim g_2$ or $h_1 \sim h_2$ — *disjunctive product* $G * H$

Finally, *strong product* $G \boxtimes H := (G \times H) \cup (G \square H)$

Products and χ

$$G \to G \,\square\, H$$

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Observation

$\chi(G \,\square\, H) \geq \max\{\chi(G), \chi(H)\}$

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Theorem (Sabidussi 1964)

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$$G \times H \to G$$

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$$\chi(G \times H) \le \min\{\chi(G), \chi(H)\}$$

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Conjecture (Hedetniemi 1966)

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Conjecture (Hedetniemi 1966)

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}$$

Theorem (Zhu 2011)

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$$

Strict vector coloring – definition

strict vector k-coloring of a graph G is $\varphi : V(G) \to$ unit vectors such that

$$u \sim v \Rightarrow \varphi(u) \cdot \varphi(v) = -\frac{1}{k-1}$$

strict vector chromatic number of a graph G

$$\bar{\vartheta}(G) = \min\{k > 1 \mid \exists \text{strict vector } k\text{-coloring of } G\}$$

- defined by [KMS 1998] to approximate $\chi(G)$
- can be approximated with arb. precision by SDP
- $\omega(G) \leq \bar{\vartheta}(G) \leq \chi(G)$ (Sandwich theorem) [GLSch 1981]
- equal to $\vartheta(\overline{G})$ defined by [Lovász 1979] to count $\Theta(C_5)$

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Lemma (Godsil, Roberson, Severini, S. 2013)

If a graph has a strict vector k-coloring then it has also a strict vector k'-coloring for every $k' > k$.

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If a graph has a strict vector k-coloring then it has also a strict vector k'-coloring for every $k' > k$.

Proof: Add a new coordinate – the value will be the same for all vertices.

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- $\geq$ holds for every monotone graph parameter
- $\leq$ needs to show: if G, H have strict vector k-colorings g, h then $G \,\square\, H$ also has a strict vector k-coloring.
- Take $g \otimes h$: put $(g \otimes h)(u, v) = g(u) \otimes h(v)$, where $u \in V(G)$ and $v \in V(H)$.

Strict vector coloring – union

- [Lovász 1979] $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$
- [Knuth 1994] $\vartheta(G * H) = \vartheta(G)\vartheta(H)$
  (observe that $G \boxtimes H \subseteq G * H$)
- observe that $\overline{G \boxtimes H} = \overline{G} * \overline{H}$ and $\overline{G * H} = \overline{G} \boxtimes \overline{H}$
- $\bar{\vartheta}(G * H) = \bar{\vartheta}(G \boxtimes H) = \bar{\vartheta}(G)\bar{\vartheta}(H)$
- $\bar{\vartheta}(G \cup H) \leq \bar{\vartheta}(G)\bar{\vartheta}(H)$
  Proof: We may assume $V(G) = V(H)$.
  $G \cup H$ is a subgraph of $G * H$ (a diagonal).

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$\bar{\vartheta}(G \times H) = \min\{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$

Proof:

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Vector coloring – definition

strict vector k-coloring of a graph G — $\varphi : V(G) \to$ unit vectors such that

$$u \sim v \Rightarrow \varphi(u) \cdot \varphi(v) = -\frac{1}{k-1}$$

strict vector chromatic number of a graph G

$$\bar{\vartheta}(G) = \min\{k > 1 \mid \exists \text{strict vector } k\text{-coloring of } G\}$$

- analogy with circular chromatic number “adjacent vertices are mapped far apart”
- this is the version originally (and mainly) considered by [KMS 1998].

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Vector coloring – union

$$\chi_v(G \cup H) \leq \chi_v(G)\chi_v(H)$$

NOT TRUE IN GENERAL [Schrijver 1979]

Vector coloring – Hedetniemi

Conjecture (Godsil, Roberson, Severini, S. 2013)

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Vector coloring for 1-homogeneous graphs

Def. A graph is *1-homogeneous* if for every $k \in \mathbb{Z}$

1. # closed k-walks in G from vertex u is independent of u **AND**
2. # k-walks in G from u to an adjacent vertex v is independent of the edge uv.

vertex-transitive and edge-transitive $\Rightarrow$ 1-homogeneous
distance-regular $\Rightarrow$ 1-homogeneous
1-homogeneous $\Rightarrow$ regular

Lemma (Godsil, Roberson, Severini, S. 2013)

If G is 1-homogeneous with degree Δ and least eigenvalue λ_{min}, then

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If G and H are 1-homogeneous, then

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Quantum coloring – motivation

- quantum theory is weird
- in order to study computational consequences, quantum information protocols/games are studied and compared with the classical setting
- one of them is quantum coloring

Quantum coloring – definition

- Game for Alice and Bob against a referee.
- Both **Alice and Bob know a graph** G and can agree on a strategy how to pretend a k-coloring of G. After that, **they may not communicate.**
- Referee chooses vertices $a, b \in V(G)$ and gives a to Alice and b to Bob.
- Alice and Bob respond with a color in $\{1, \ldots, k\}$ — **"pretending this is the color of their vertex"**
- If $a = b$, the color must be the same, if $a \sim b$, it must be different.
- Alice and Bob only care about **100%-proof strategies.**

Quantum coloring – definition

- Rather obviously, Alice and Bob win iff $k \geq \chi(G)$.
- However, by sharing a *quantum entanglement* they may win for smaller k's.

$$\chi_q(G) := \min\{k : \text{A \& B can win}\}$$

- For *Hadamard* graphs Ω_{4n} the separation is exponential
- $\chi_q(G) \leq k \Leftrightarrow G$ has a *quantum homomorphism* to K_k $\Leftrightarrow G \to M(K_k)$ (for a certain graph $M(K_k)$). [Mančinska, Roberson 2012]

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Vector chromatic theory

Find nice theorems for χ_v, $\bar{\vartheta}$, χ_q **as chromatic-type numbers.**