On the directed Oberwolfach Problem with equal cycle lengths

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Outline

- Introduction: The directed Oberwolfach Problem.
- Main results.
- Terminology.
- Tools.
- Ideas from proofs.

Resolvable directed cycle systems

— the directed Oberwolfach Problem with equal cycle lengths

- **Directed Oberwolfach Problem with equal cycle lengths:** Determine the necessary and sufficient conditions on n and m for there to exist a decomposition of K_n^* into spanning subdigraphs, each a disjoint union of directed m-cycles (that is, a $RCS^*(m,n)$).
- Obvious **necessary condition**: $m|n$.
- **Previous results:**

Theorem (Bermond, Germa, Sotteau, 1979)

There exists a $RCS^*(3,n)$ *if and only if* $3|n$ *and* $n \neq 6$.

Theorem (Bennett, Zhang, 1990)

There exists a $RCS^*(4,n)$ *if and only if* $4|n$ *and* $n \neq 4$.

New results

Theorem (Burgess, Francetić, Niesink, Šajna)

There exist the following:

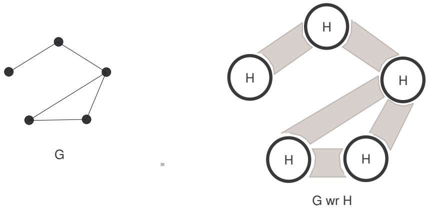
- *a* $RCS^*(m, \alpha m)$ *for odd* $m \geq 5$ *and odd* α;
- *a* $RCS^*(m, \alpha m)$ *for even* $m \geq 6$ *and all* $\alpha \geq 2$;
- *a* $RCS^*(m, \alpha m)$ *for odd* m, $7 \leq m \leq 49$, $3 \not| m$, *and all even* α;
- *a* $RCS^*(m, \alpha m)$ *for* $m = 9, 15, 21$, *and all even* α.

Lemma

If there exists a $RCS^*(m, 2m)$, *then there exists a* $RCS^*(m, \alpha m)$ *for all even* α.

Basic terminology

- G^*: the (symmetric) digraph obtained from a graph G by replacing each edge uv with the two arcs (u, v) and (v, u)
- K_n^*, $K_{m,n}^*$
- C_m and $\vec{C}_m$: cycle and directed cycle of length m
- **Wreath product** $G \star H$ of (di)graphs G and H:
  obtained from G by replacing every vertex u of G with a copy H_u of H, and for each edge uv (arc (u, v)) of G, inserting an edge (arc) from every vertex of H_u to every vertex of H_v



Terminology

- **Decomposition** $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$:
  partition of $E(G)$ into $E(H_1), E(H_2), \ldots, E(H_k)$, for subgraphs $H_1, H_2, \ldots, H_k$ of G
- H-**decomposition** of a graph G:
  decomposition of G into copies of a subgraph H
- **Resolution class:**
  a subset $\{H_{i_1}, H_{i_2}, \ldots, H_{i_t}\}$ of a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k\}$ of G such that $\{V(H_{i_1}), V(H_{i_2}), \ldots, V(H_{i_t})\}$ is a partition of $V(G)$
- **Resolvable decomposition:**
  a decomposition that can be partitioned into resolution classes
- $RCS(m, G)$: resolvable m-cycle decomposition of a graph G
- $CS^*(m, D)$: directed m-cycle decomposition of a digraph D
- $RCS^*(m, D)$: resolvable dir. m-cycle decomposition of a digraph D

Tools: previous results

Theorem (Alspach, Schellenberg, Stinson, Wagner)

There exists a $RCS(m, K_n)$ *if and only if* n *is odd and* $m|n$.

Theorem (Alspach, Jordon, Šajna, Verrall)

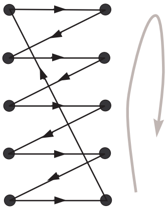
There exists a $CS^*(m, K_n^*)$ *if and only if* $m|n(n-1)$ *and* $(m, n) \notin \{(4,4), (3,6), (6,6)\}$.

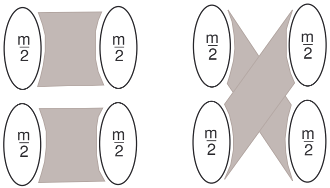
Theorem (Liu)

There exists a $RCS(m, K_n \star \bar{K}_t)$ *if and only if* $m|nt$, $t(n-1)$ *is even,* m *is even if* $n = 2$, *and* $(m, n, t) \notin \{(3,3,2), (3,3,6), (3,6,2), (6,2,6)\}$.

Tools: small constructions 1

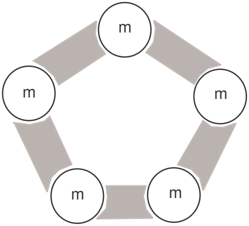
For even $m \geq 4$, there exist a $CS^*(m, K^*_{\frac{m}{2},\frac{m}{2}})$ and $RCS^*(m, K^*_{m,m})$.

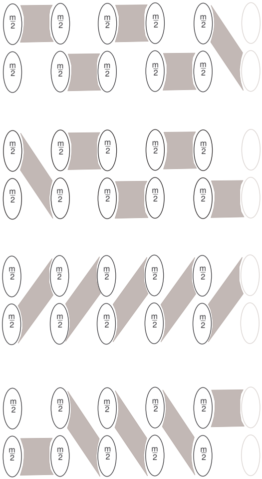




Tools: small constructions 2

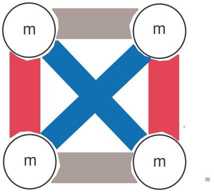
- Let $m \geq 4$ be even and $\alpha \geq 3$.
- There exists a $RCS^*(m, C_\alpha^* \star \bar{K_m})$.
- Case α odd:

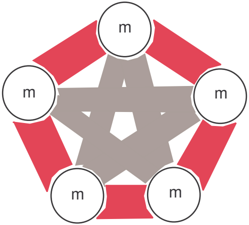




Tools: small constructions 3

- Let $m \geq 4$ be even and $\alpha \geq 2$.
- Case α even: there exists a 1-factorization of K_α.
- Case α odd: there exists a $RCS(\alpha, K_\alpha)$.
- Hence there exists a $RCS^*(m, K_\alpha^* \star \bar{K}_m)$.





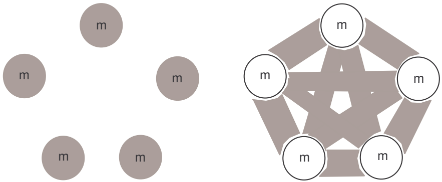
Proof ideas: Case α and m both odd, or $m \geq 8$ even.

Case α and m both odd.

- By [ASSW], there exists a $RCS(m, K_{\alpha m})$.
- Direct each cycle in this decomposition once in each possible direction to obtain a $RCS^*(m, K^*_{\alpha m})$.

Case $m \geq 8$ even.

- Decompose $K^*_{\alpha m} = \bar{K}_\alpha \star K^*_m \oplus K^*_\alpha \star \bar{K}_m$.
- There exists a $CS^*(m, K^*_m)$ by [AJŠV].
- Hence there exists a $RCS^*(m, \bar{K}_\alpha \star K^*_m)$.
- There exists a $RCS^*(m, K^*_\alpha \star \bar{K}_m)$ as seen before (also by [Liu]).

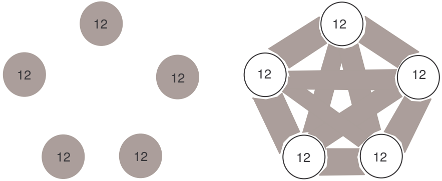


Proof ideas: Case $m = 6$.

Challenge: a $CS^*(6, K_6^*)$ does not exist.

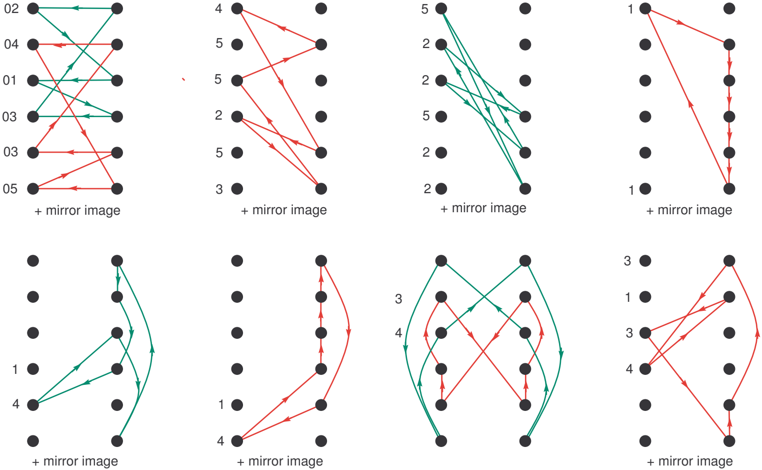
Case $\alpha \geq 2$ even.

- Decompose $K_{6\alpha}^* = \bar{K}_{\frac{\alpha}{2}} \star K_{12}^* \oplus K_{\frac{\alpha}{2}}^* \star \bar{K}_{12}$.
- There exists a $RCS^*(6, K_{\frac{\alpha}{2}}^* \star \bar{K}_{12})$ by [Liu].
- There exists a $RCS^*(6, K_{12}^*)$ (shown on the next page).



Proof ideas: Case $m = 6$ — continued.

A $RCS^*(6, K^*_{12})$:

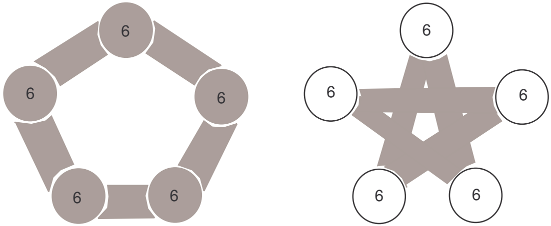


+ 2 more resolution classes

Proof ideas: Case $m = 6$ — continued.

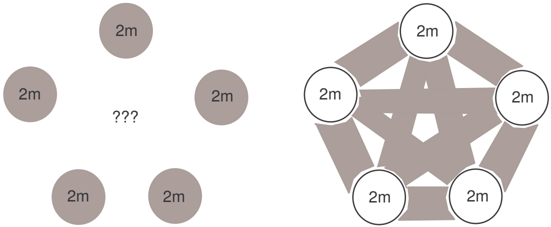
Case $\alpha \geq 3$ odd.

- There exists a $CS(\alpha, K_\alpha)$.
- Decompose $K_{6\alpha}$ into $C_\alpha^* \star K_6^*$ and $\frac{\alpha-1}{2}$ copies of $C_\alpha^* \star \bar{K}_6$.
- There exists a $RCS^*(6, C_\alpha^* \star K_6^*)$ (special construction).
- There exists a $RCS^*(6, C_\alpha^* \star \bar{K}_6)$ as seen.



Proof ideas: Case α even, m odd.

- Decompose $K^*_{\alpha m} = \bar{K}_{\frac{\alpha}{2}} \star K^*_{2m} \oplus K^*_{\frac{\alpha}{2}} \star \bar{K}_{2m}$.
- By [Liu], there exists a $RCS(m, K_{\frac{\alpha}{2}} \star \bar{K}_{2m})$.
- Hence, there exists a $RCS^*(m, K^*_{\frac{\alpha}{2}} \star \bar{K}_{2m})$.
- Then the existence of a $RCS^*(m, K^*_{2m})$ implies the existence of a $RCS^*(m, K^*_{\alpha m})$.



What we know about $RCS^*(m, 2m)$

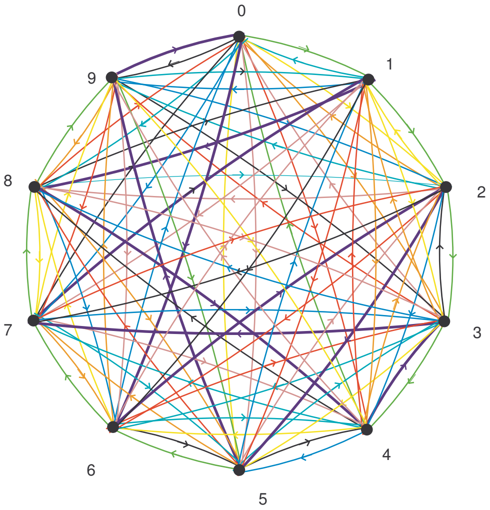
Proposition

$RCS^*(m, 2m)$ exists for

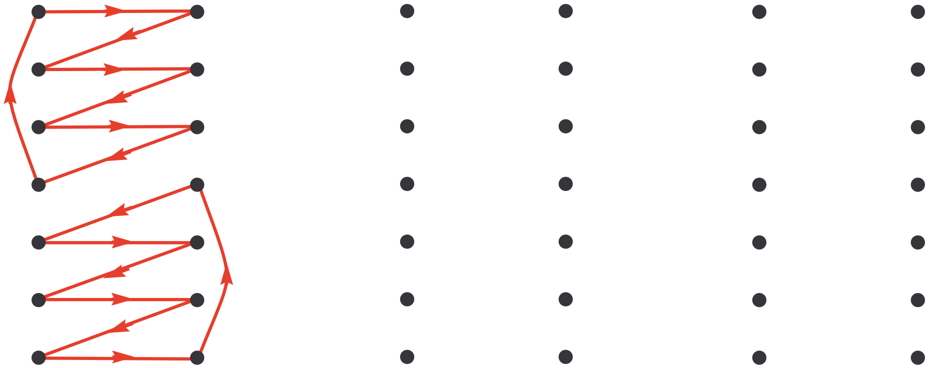
- *odd m, $7 \leq m \leq 49$, $3 \not| m$;*
- $m = 9, 15, 21$.

Proof ideas: $m = 5$

There exist a $RCS^*(5, K_{10}^*)$:



Proof ideas: A $RCS^*(m, 2m)$ for $m \equiv 1$ or $5 \pmod 6$, $7 \leq m \leq 49$.



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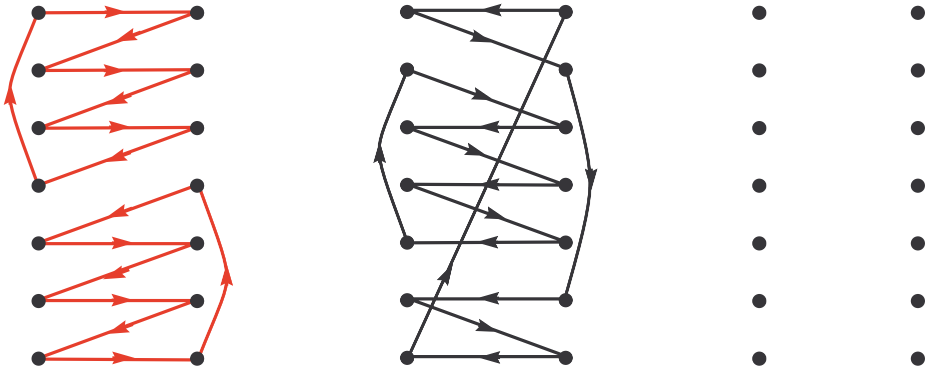

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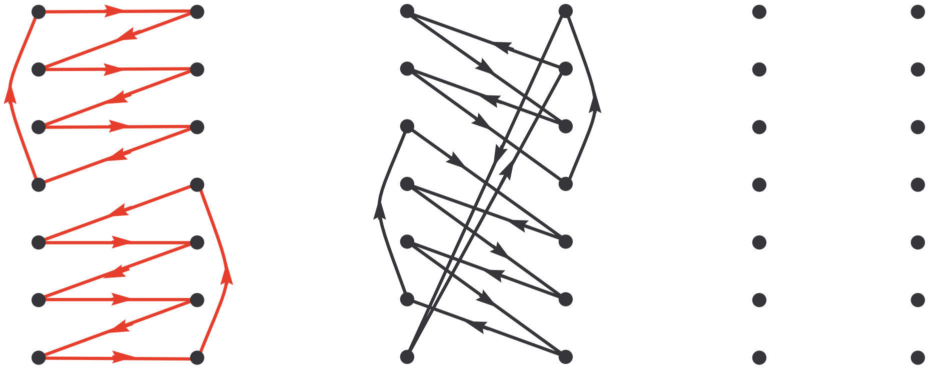

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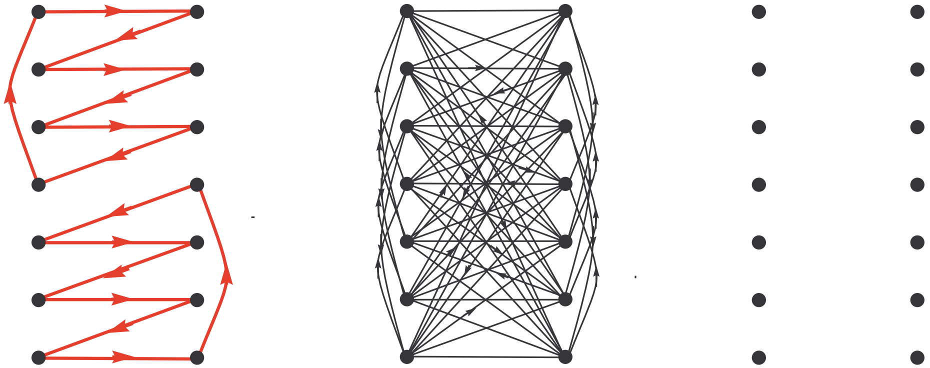

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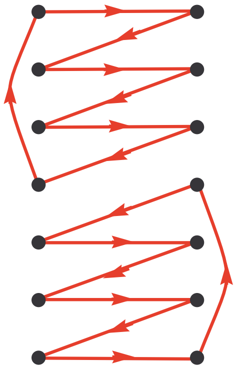

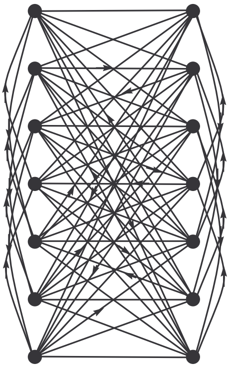

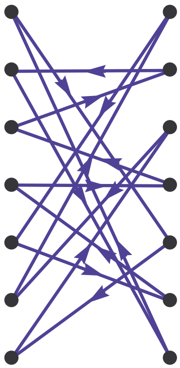

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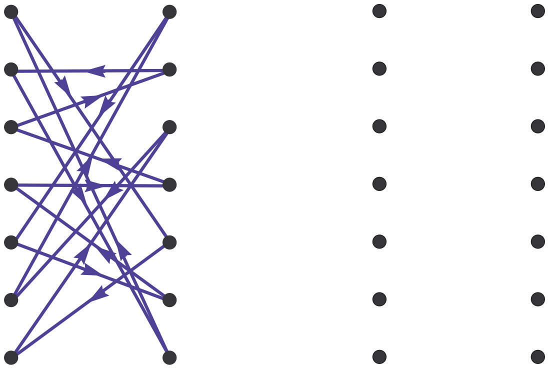

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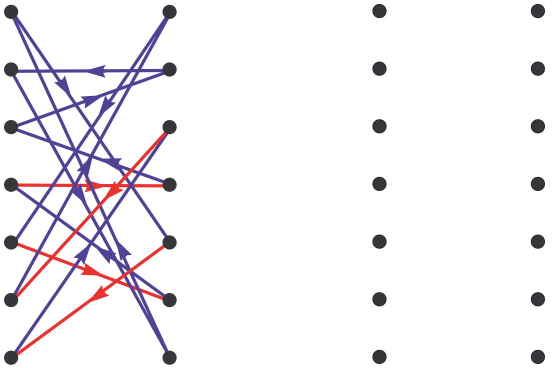

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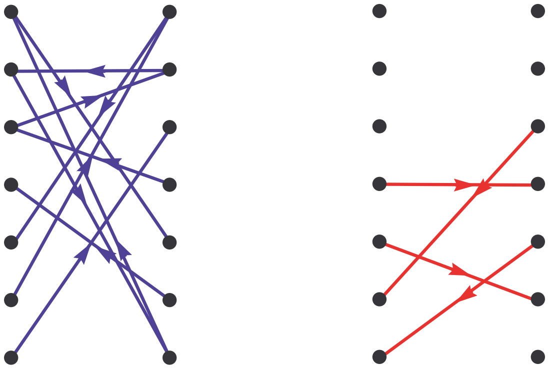

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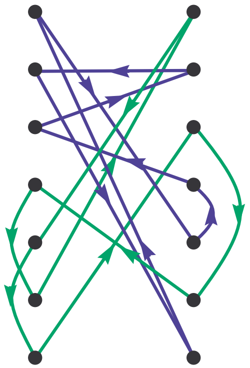

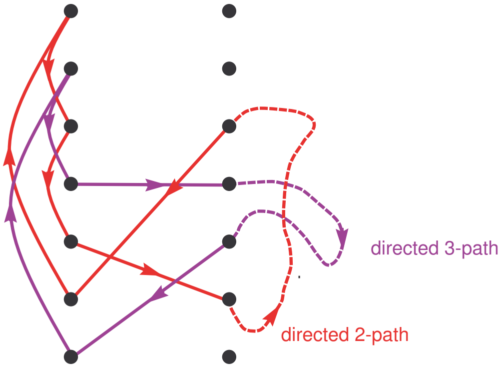

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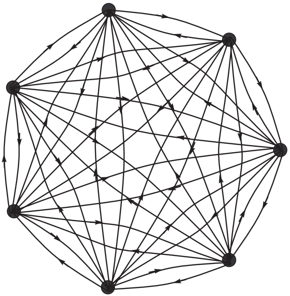

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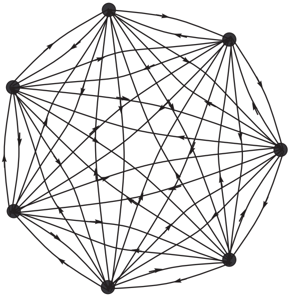

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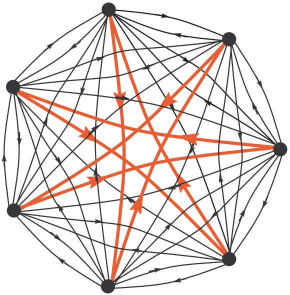

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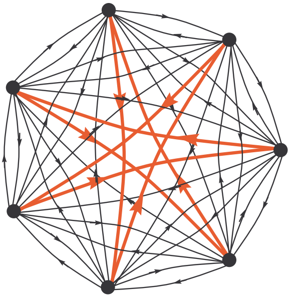

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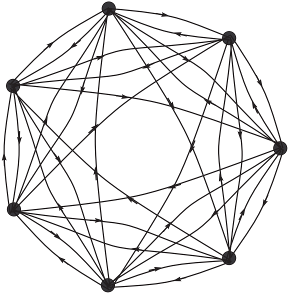

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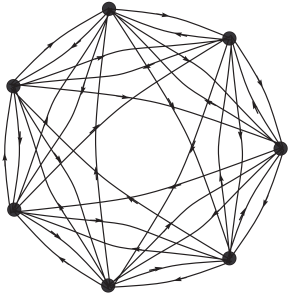

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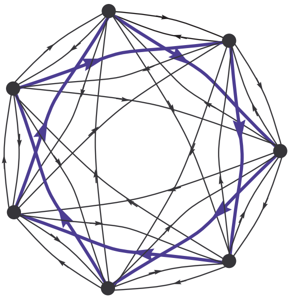

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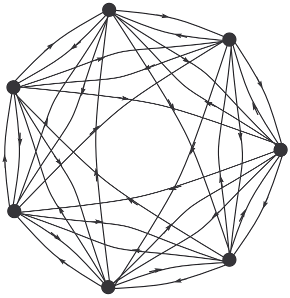

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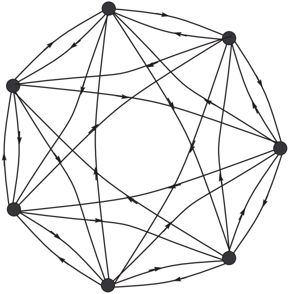

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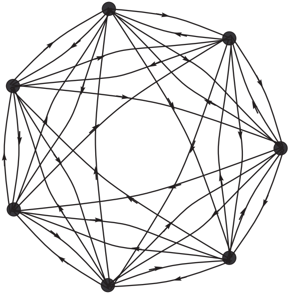

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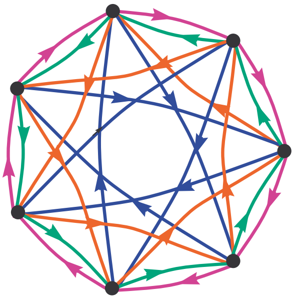

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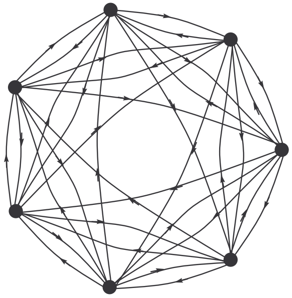

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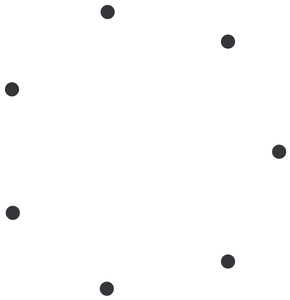

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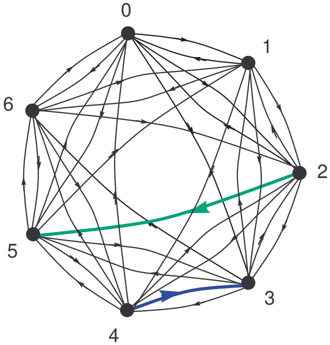

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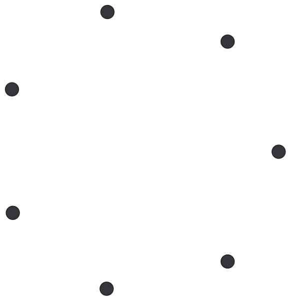

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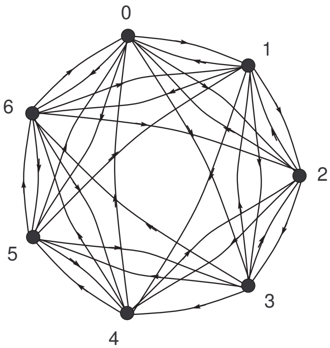
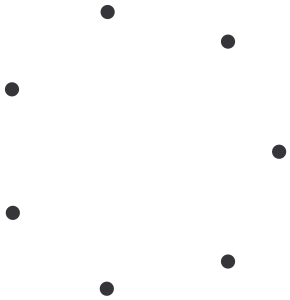


Right

It remains to decompose the digraph on the right into:

- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
- directed Hamilton cycles

Proof ideas: A $RCS^*(m, 2m)$ for $m \equiv 1$ or $5 \pmod 6$, $7 \leq m \leq 49$.



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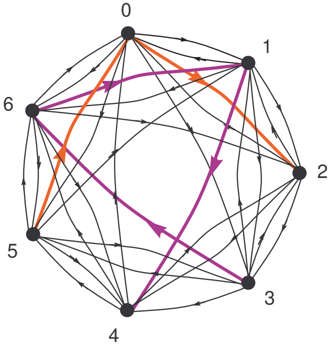
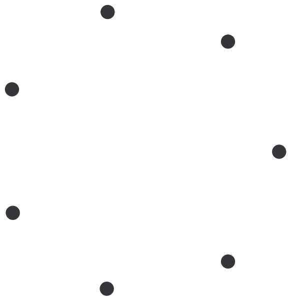


Right

It remains to decompose the digraph on the right into:

- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
- directed Hamilton cycles

Proof ideas: A $RCS^*(m, 2m)$ for $m \equiv 1$ or $5 \pmod 6$, $7 \leq m \leq 49$.



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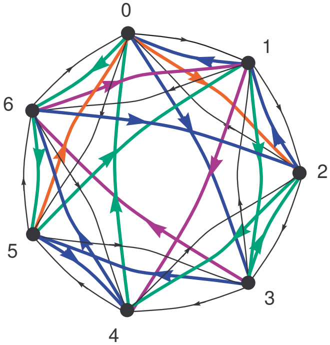

Right

It remains to decompose the digraph on the right into:

- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
- directed Hamilton cycles

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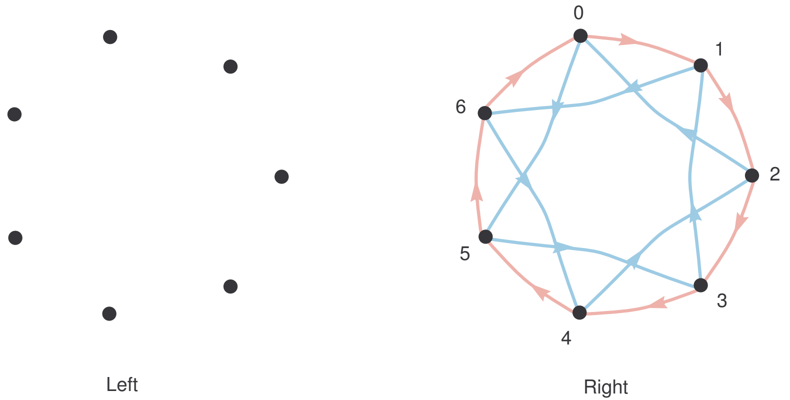

It remains to decompose the digraph on the right into:

- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
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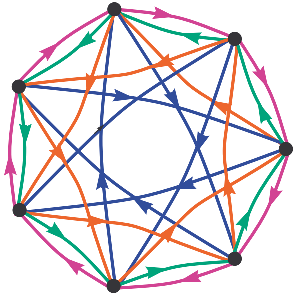
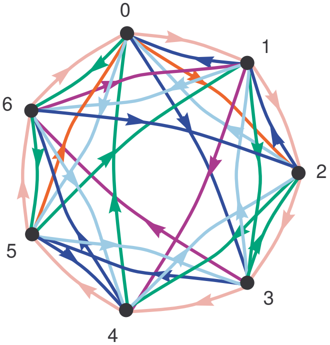


Left



Right

It remains to decompose the digraph on the right into:

- vertex-disjoint dir. $(5, 2)$-path of length 2 and $(3, 4)$-path of length 3, and
- directed Hamilton cycles

Thank you!