Eulerian-type properties of hypergraphs

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Outline

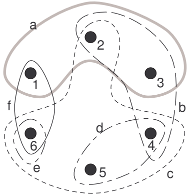
- Basic definitions.
- Walks, trails, paths, cycles
- Hypergraphs with an Euler tour
- Hypergraphs with a strict Euler tour
- Other eulerian-type properties of hypergraphs

Hypergraphs

- **Hypergraph** $H = (V, E)$:
  - vertex set $V \neq \emptyset$
  - edge set $E \subseteq 2^V - \{\emptyset\}$
  - incidence (v, e) for $v \in V$, $e \in E$, $v \in e$
- **Degree** of vertex v in H: $\deg_H(v) = |\{e \in E : v \in e\}|$
- **Size** of edge e in H: $|e|$
- r-**regular hypergraph**: $\deg_H(v) = r$ for all $v \in V$
- k-**uniform hypergraph**: $|e| = k$ for all $e \in E$

NOTE: A 2-uniform hypergraph is a *simple* graph.

EXAMPLE: A hypergraph $H = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{a, b, c, d, e, f\}$.



New hypergraphs from old

- **Subhypergraph** H' of $H = (V, E)$:
  - $H' = (V', E')$ is a hypergraph
  - $V' \subseteq V$
  - $E' \subseteq E$
- **Spanning subhypergraph**: $V' = V$
- **Vertex-induced subhypergraph** $H' = H[V']$: $E' = E \cap 2^{V'}$
- **Edge-deleted subhypergraph**: $H - e = (V, E - \{e\})$ for $e \in E$
- r-**factor** of H: r-regular spanning subhypergraph of H
- **Union** of hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$: $H_1 \cup H_2 = (V_1 \cup V_2, E_1 \cup E_2)$
- **Decomposition** $H = H_1 \oplus H_2$ of hypergraph H: $H = H_1 \cup H_2$ such that $E_1 \cap E_2 = \emptyset$

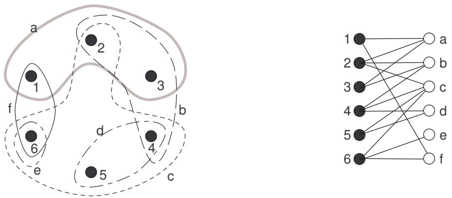
Incidence graph of a hypergraph

- **Incidence graph** $G = G(H)$ of a hypergraph $H = (V, E)$:
  - $V(G) = V \cup E$
  - $E(G) = \{ve : v \in V, e \in E, v \in e\}$

Lemma

Let $H = (V, E)$ be a hypergraph.

- *If H' is a subhypergraph of H, then $G(H')$ is a subgraph of $G(H)$.*
- *If G' is a subgraph of $G = G(H)$ such that $\deg_{G'}(e) = \deg_G(e)$ for all $e \in E$, then G' is the incidence graph of a subhypergraph of H.*



Walks in a hypergraph

- **Walk of length** $k \geq 0$ in a hypergraph $H = (V, E)$:
  $v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$ such that
  - $v_0, v_1, \ldots, v_k \in V$
  - $e_1, \ldots, e_k \in E$
  - $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, k$
  - $v_{i-1} \neq v_i$ for all $i = 1, \ldots, k$
- **Closed walk**: $v_0 = v_k$
- **Hypergraph H' associated with the walk W**:
  - $V(H') = \bigcup_{i=1}^{k} e_i$
  - $E(H') = \{e_1, \ldots, e_k\}$
- **Anchors** of the walk W: $v_0, v_1, \ldots, v_k$
- **Floaters** of the walk W: vertices in $V(H') - \{v_0, v_1, \ldots, v_k\}$

Walks, trails, paths, cycles

A walk $W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ is called a

- **trail** if the anchor incidences
  $(v_0, e_1), (v_1, e_1), (v_1, e_2), \dots, (v_k, e_k)$
  are pairwise distinct
- **strict trail** if it is a trail and the edges $e_1, \dots, e_k$ are pairwise distinct
- **weak path** if it is a trail and the vertices $v_0, v_1, \dots, v_k$ are pairwise distinct (but edges may not be)
- **path** if both the vertices $v_0, v_1, \dots, v_k$ and the edges $e_1, \dots, e_k$ are pairwise distinct
- **cycle** if W is a closed walk and both the vertices $v_0, v_1, \dots, v_{k-1}$ and the edges $e_1, \dots, e_k$ are pairwise distinct

Similarly we define a **closed trail, strict closed trail, weak cycle**.

Walks in a hypergraph and its incidence graph

Lemma

Let $H = (V, E)$ be a hypergraph and $G = G(H)$ its incidence graph. Consider

$$W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k.$$

Then:

- *W is a walk in H if and only if W is a walk in G.*
- *W is a trail/path/cycle in H if and only if W is a trail/path/cycle in G.*
- *W is a strict trail in H if and only if W is a trail in G that visits every $e \in E$ at most once.*
- *W is a weak path/weak cycle in H if and only if W is a trail/closed trail in G that visits every $v \in V$ at most once.*

Connectedness

- **Connected hypergraph** $H = (V, E)$: there exists a (u, v)-walk (equivalently (u, v)-path) for all $u, v \in V$
- **Connected component** of H: maximal connected subhypergraph of H
- $\omega(H)$ = number of connected components of H

Corollary

A hypergraph is connected if and only if its incidence graph is connected.

First generalization: Euler tours

- **Euler tour** of a hypergraph H: closed trail of H containing every incidence of H

Theorem

A connected hypergraph $H = (V, E)$ has an Euler tour if and only if its incidence graph $G(H)$ has an Euler tour, that is, if and only if $\deg_H(v)$ and $|e|$ are even for all $v \in V$, $e \in E$.

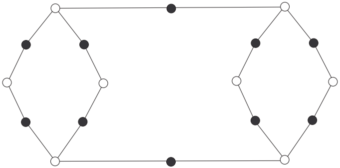
Corollary

Let $H = (V, E)$ be a connected hypergraph such that $\deg_H(v)$ and $|e|$ are even for all $v \in V$, $e \in E$. Then H is a union of hypergraphs associated with cycles that are pairwise anchor incidence-disjoint.

Second generalization: strict Euler tours and Euler families

- **Strict Euler tour** of a hypergraph H: strict closed trail of H containing every edge of H
- **Euler family** of a hypergraph H: a family of strict closed trails of H that are pairwise anchor-disjoint such that each edge of H lies in exactly one trail

EXAMPLE: Incidence graph of a connected hypergraph with an Euler family but no strict Euler tour.



Characterizing hypergraphs with strict Euler tours – 1

Theorem

A hypergraph H has an Euler family (strict Euler tour) if and only if its incidence graph $G(H)$ has a (connected) subgraph G' such that $\deg_{G'}(e) = 2$ for all $e \in E$ and $\deg_{G'}(v)$ is even for all $v \in V$.

Some sufficient conditions:

Corollary

Let H be a hypergraph with the incidence graph $G = G(H)$. If G has a 2-factor, then H has an Euler family. If G is hamiltonian, then H has a strict Euler tour.

Corollary

Let H be an r-regular r-uniform hypergraph for $r \geq 2$. Then H has an Euler family.

Characterizing hypergraphs with strict Euler tours – 2

Theorem (Lonc and Naroski, 2010)

The problem of determining whether a given k-uniform hypergraph has a strict Euler tour is NP-complete for $k \geq 3$.

Lemma (Lonc and Naroski, 2010)

If a hypergraph $H = (V, E)$ has a strict Euler tour, then $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$.

Is this condition also sufficient (for connected hypergraphs)?

- Yes! for connected graphs
- Yes! for certain uniform hypergraphs

Characterizing hypergraphs with strict Euler tours – 3

Theorem (Lonc and Naroski, 2010)

A k-uniform hypergraph $H = (V, E)$ with a connected strong connectivity graph has a strict Euler tour if and only if $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$.

- **Strong connectivity graph** G of a k-uniform hypergraph $H = (V, E)$:
  - $V(G) = E$
  - $E(G) = \{ef : e, f \in E, |e \cap f| = k - 1\}$

Characterizing hypergraphs with strict Euler tours – 4

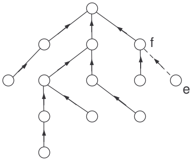
Theorem

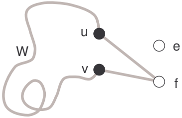
Let $H = (V, E)$ be a hypergraph such that its strong connectivity digraph has a spanning arborescence. Then H has a strict Euler tour if and only if $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$.

- **Strong connectivity digraph** D_c of a hypergraph $H = (V, E)$:
  - $V(D_c) = E$
  - $A(D_c) = \{(e, f) : e, f \in E, |f - e| = 1, |e \cap f| \geq 3\}$.
- **Spanning arborescence** of a digraph D: spanning subdigraph that is a directed tree with all arcs directed towards the root

Proof of sufficiency

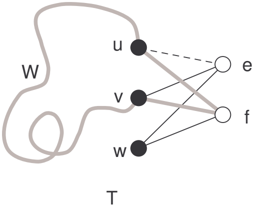
- Assume $\sum_{v\in V}\lfloor\frac{\deg_H(v)}{2}\rfloor \geq |E|$. Then $|E|\geq 2$. By induction on $|E|$.
- Suppose $E=\{e,f\}$. Since $D_c(H)$ has a spanning arborescence, $|e\cap f|\geq 3$. If $\{u,v\}\subseteq e\cap f$, $u\neq v$, then $T=uevfu$ is a strict Euler tour of H.
- Let $H=(V,E)$ be a hypergraph with $|E|\geq 3$ such that its strong connectivity digraph $D_c(H)$ has a spanning arborescence A.
- Let $e\in E$ be a leaf of A and f its outneighbour in A. Then $|f-e|=1$ and $|e\cap f|\geq 3$. Since $D_c(H-e)$ has a spanning arborescence $A-e$, the hypergraph $H-e$ has a strict Euler tour $T=ufvW$ (where W is an appropriate (v,u)-walk).

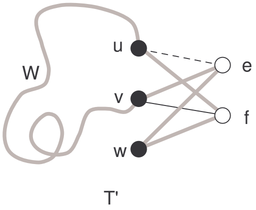




Proof of sufficiency – cont'd

- Since $|f - e| = 1$, at least on of u, v is in e; say v.
- Since $|e \cap f| \geq 3$, there exists $w \in e \cap f$, $w \neq u, v$.
- Then $T' = ufwevW$ is a strict Euler tour of H.





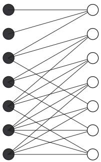
Counterexample 1: vertices of degree 1

Lemma

Let $H = (V, E)$ be a hypergraph with an edge $e = \{v_1, v_2, \dots, v_k\}$ such that $\deg(v_i) = 1$ for all $i = 1, 2, \dots, k-1$. Then H has no strict Euler tour.

Hypergraph H (incidence graph shown below):

- connected
- satisfies the necessary condition $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$
- has no Euler family



Cut edges

- **Cut edge** in a hypergraph $H = (V, E)$: edge $e \in E$ such that $\omega(H - e) > \omega(H)$.

Lemma

Let $H = (V, E)$ be a hypergraph with a cut edge. Then H has no strict Euler tour.

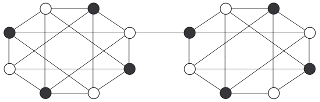
Lemma

Let $H = (V, E)$ be a hypergraph and $e \in E$. Then e is a cut edge of H if and only if it is a cut vertex of $G(H)$.

Counterexample 2: cut edges

Hypergraph H (incidence graph is shown below):

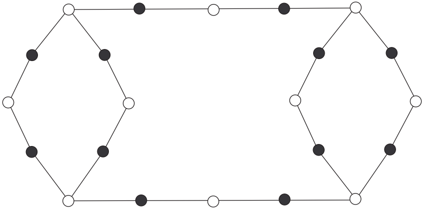
- connected
- satisfies the necessary condition $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$
- has no Euler family



Counterexample 3: no cut edges

Hypergraph H (incidence graph is shown below):

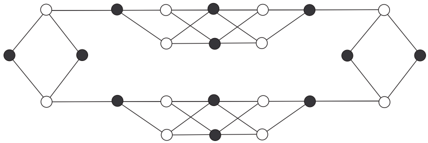
- connected
- has no cut edges
- satisfies the necessary condition $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$
- has no Euler family



Counterexample 4: a uniform hypergraph without cut edges

Hypergraph H (incidence graph is shown below):

- 3-uniform
- connected
- has no cut edges
- satisfies the necessary condition $\sum_{v \in V} \lfloor \frac{\deg_H(v)}{2} \rfloor \geq |E|$
- has no Euler tour



Dual hypergraph

- **Dual** H^* of a hypergraph $H = (V, E)$:
  - hypergraph $H^* = (E, V^*)$
  - $V^* = \{v^* : v \in V\}$ where $v^* = \{e \in E : v \in e\}$

EXAMPLE: The incidence graph of a hypergraph and its dual.

Lemma

A hypergraph H is 2-regular if and only if its dual is a simple graph.

2-factors and Euler families

Theorem

Let $H = (V, E)$ be a hypergraph such that every edge of H has even size.

- *If H has a 2-factor, then its dual H^* has an Euler family.*
- *If H has a connected 2-factor, then its dual H^* has a strict Euler tour.*

Theorem

Let $H = (V, E)$ be a hypergraph and H^* its dual. Suppose H has an Euler family $\mathcal{F}$ with the property that, for any vertex $u \in V$, if u is an anchor of a strict closed trail in $\mathcal{F}$, then every incidence (u, e) is traversed by $\mathcal{F}$. Then H^* has a 2-factor.

Block decomposition of hypergraphs and strict Euler tours

- **Separating vertex** in a hypergraph H: vertex v such that H decomposes into two subhypergraphs with exactly v in common
- **Non-separable hypergraph**: hypergraph with no separating vertices
- **Block** of a hypergraph H: maximal non-separable subhypergraph of H

Lemma

- *Any two distinct blocks of a hypergraph H have at most one vertex in common.*
- *The blocks of H form a decomposition of H.*

Theorem

- *A hypergraph H has an Euler family if and only if each block of H has an Euler family.*
- *H has a strict Euler tour (necessarily traversing every separating vertex of H) if and only if each block B of H has a strict Euler tour that traverses every separating vertex of H contained in B.*

Cycle decomposition of hypergraphs

RECALL: A connected graph admits a cycle decomposition if and only if it has an Euler tour. True for hypergraphs?

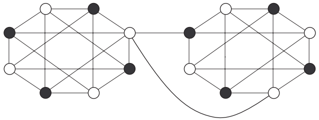
Theorem

A connected hypergraph admits a cycle decomposition if and only if it has an Euler family.

More on cut edges

RECALL: An even graph has no cut edges. True for hypergraphs?

COUNTEREXAMPLE: An even hypergraph with a cut edge.



Analogue for hypergraphs:

Lemma

Let $H = (V, E)$ be a k-uniform hypergraph such that $\deg(u) \equiv 0 \pmod{k}$ for all $u \in V$. Then H has no cut edges.

Strong cut edges

RECALL: An edge of a graph is a cut edge if and only if it lies in no cycle. True for hypergraphs?

- **Strong cut edge** of H: cut edge e of H such that $\omega(H - e) = \omega(H) + |e| - 1$.

Lemma

Let e be an edge in a hypergraph H. Then e is a strong cut edge if and only if it lies in no cycle of H.

Lemma

Let $H = (V, E)$ be a hypergraph such that $\deg_H(v)$ and $|e|$ are even for all $v \in V$, $e \in E$. Then H has no strong cut edges.

Thank you!