Spectrum of packing and covering of the complete graph with stars

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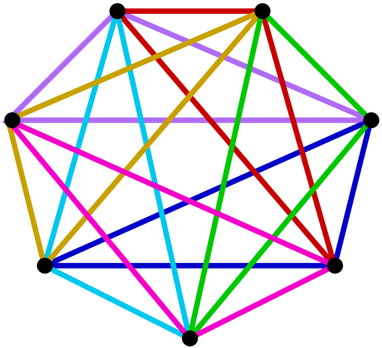
Definitions and history

For any simple graph G, a G-decomposition of a simple graph H is a partition of the edge set of H with graphs all isomorphic to G. If H is the complete graph K_n, then the decomposition is called a G-design of order n.

Example. K_3-design of order 7.



Definitions and history

The spectrum problem for a graph G is to determine the set D of all positive integers n such that a G-design of order n exists if and only if $n \in D$.

In 1972, Hell and Rosa introduced G-designs in order to attack the spectrum problem for P_3, a path on three vertices.

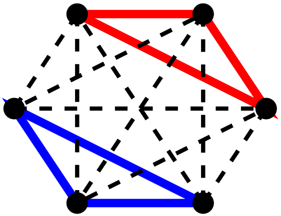
The spectrum problem has been considered for many classes of graphs (*A survey on the existence of G-designs*, Adams, P.; Bryant, D.; and Buchanan, M., 2008)

Definitions and history

What if the decomposition does not exist?

A set of subgraphs of H such that each subgraph is isomorphic to G and every edge of H is contained in **at most** one subgraph is a G-packing of H. The leave graph is the graph consisted of those edges of H which are included in none of these subgraphs.

Example. A K_3-packing of K_6



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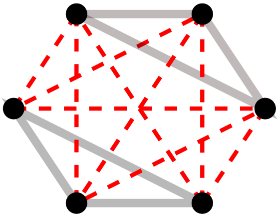
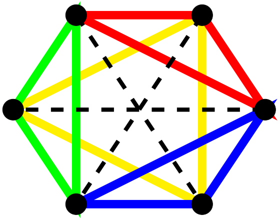


Definitions and history

A G-packing is a maximum G-packing if it has the smallest possible number of edges in the leave graph.

Example. A maximum K_3-packing of K_6 with the leave $3K_2$



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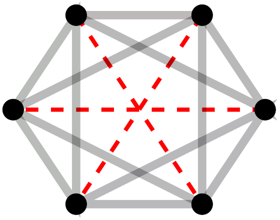

Definitions and history

A set of subgraphs of H such that each subgraph is isomorphic to G and every edge of H is contained in **at least** one subgraph is a G-covering of H. For any G-covering, C, of H, the excess graph is the multigraph $\mathcal{C} \setminus H$ where $\mathcal{C}$ is the graph obtained from the union of all the subgraphs in the covering C.

Definitions and history

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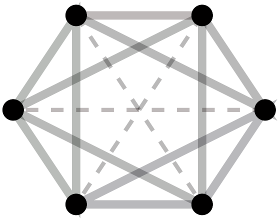

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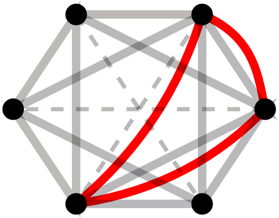

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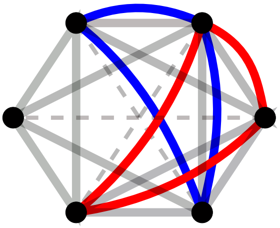

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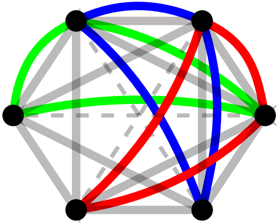

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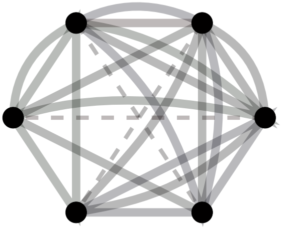

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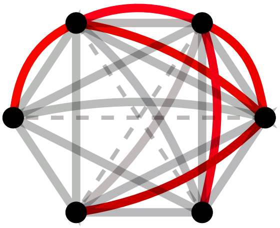

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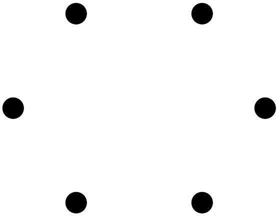
A G-covering is called a minimum G-covering if it has the smallest possible number of edges in the excess graph.

Example. A minimum K_3-covering of K_6 with the excess $3K_2$



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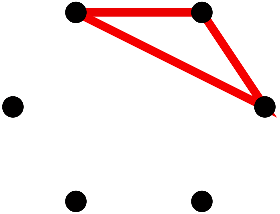

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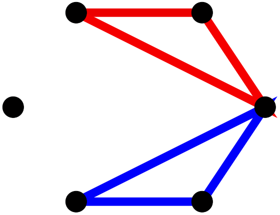

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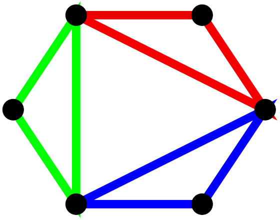

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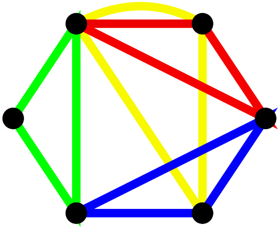

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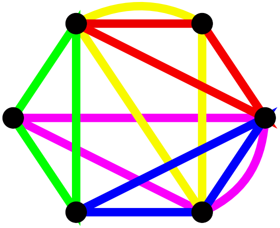

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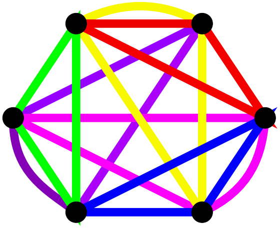

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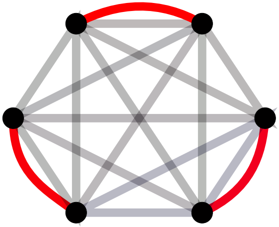


Definitions and history

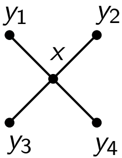
The packing problem (covering problem) for a graph G is to determine the number of elements in a maximum G-packing (minimum G-covering) of K_n and this number is called the G-packing number (G-covering number).

In 1999, Adams, Bryant, and El-Zanati solved the packing and covering problems for 3-cubes. In 2008, Bryant and Horsley found sufficient conditions for the existence of a packing of the complete graph with cycles of specific lengths. Bryant also proved Tarsi's conjecture on necessary and sufficient conditions for the existence of a packing of the complete multigraph with paths of specific lengths in 2010.

Definitions and history

The complete bipartite graph $K_{1,n}$ is called an *n*-star and is denoted by S_n.

Example. A 4-star



Theorem [Yamamoto et. al. 1975]

For $k \geq 1$, K_n has an S_k-decomposition if and only if $n = 1$, or $n \geq 2k$ and $n(n-1) \equiv 0 \pmod{2k}$.

Definitions and history

The packing and covering problems were solved for all trees of order seven or less by Roditty in 1983, 1985, 1986, and 1993. In particular, he solved the problems for stars with up to six edges.

Theorem [Roditty, 1983, 1985, 1986, 1993]

If n and k are integers such that $n \geq 2k-1$ and $k \leq 6$, then the S_k-packing number of the complete graph K_n is $\lfloor \frac{n(n-1)}{2k} \rfloor$ and if $n \geq 2k$ and $k \leq 6$, then the S_k-covering number of K_n is $\lceil \frac{n(n-1)}{2k} \rceil$.

However, Roditty did not achieve all the possible leaves and excesses, which we refer to as the spectrum problem for packing and covering.

Our work

Theorem 1. Let $k \leq 5$, $n \geq 2k - 1$ be an integer and the leave graph in a maximum S_k-packing of the complete graph K_n have i edges. For any graph H with i edges there exists a maximum S_k-packing of K_n with H as the leave graph.

Theorem 2. Let $k \leq 5$, $n \geq 2k$ and the excess graph in a minimum S_k-covering of the complete graph K_n have i edges. For any graph H with i edges there exists a minimum S_k-covering of K_n with H as the excess graph.

Our work

Lemma

If k is a positive odd integer, $n \geq \frac{k+1}{2}$ is an integer, and H is the leave graph (excess graph) in an S_k-packing (S_k-covering) of the complete graph K_n, then there exists an S_k-packing (S_k-covering) of K_{n+k} with H as the leave graph (excess graph).

For odd k and large enough n,

$$S_k\text{-packing (covering) of } K_n\text{, leave (excess) } H$$
$$\downarrow$$
$$S_k\text{-packing (covering) of } K_{n+k}\text{, leave (excess) } H$$

Our work

In order to prove our main theorems, we consider congruence classes mod $2k$.

For $k = 5$, from Yamamoto's theorem we know that decompositions exist for $n \equiv 0, 1, 5,$ and $6 \pmod{10}$. We need to show that for all remaining congruence classes, we can achieve all possible leaves and excesses. By Lemma 1, it suffices to achieve all leaves and excesses for $n = 12, 13, 14$.

Let $n = 14$. Since $\frac{n(n-1)}{2} = \frac{14 \cdot 13}{2} \equiv 1 \pmod{5}$, the leave graph in any maximum S_5-packing of K_{14} has a single edge and hence, the excess graph in any minimum S_5-covering of K_{14} has 4 edges.

Our work

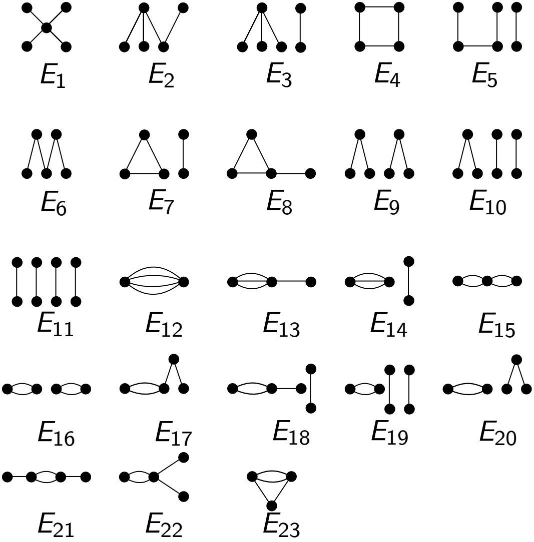
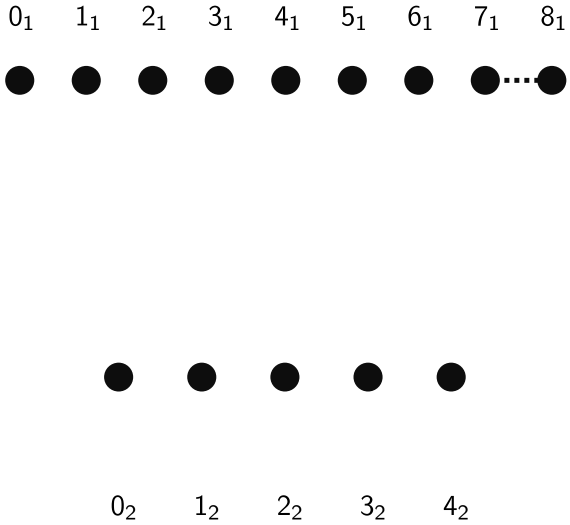


Figure : All possible 4-edge excesses

Our work



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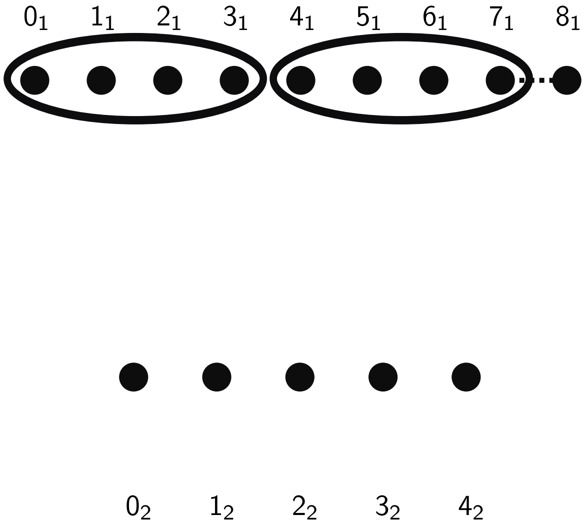

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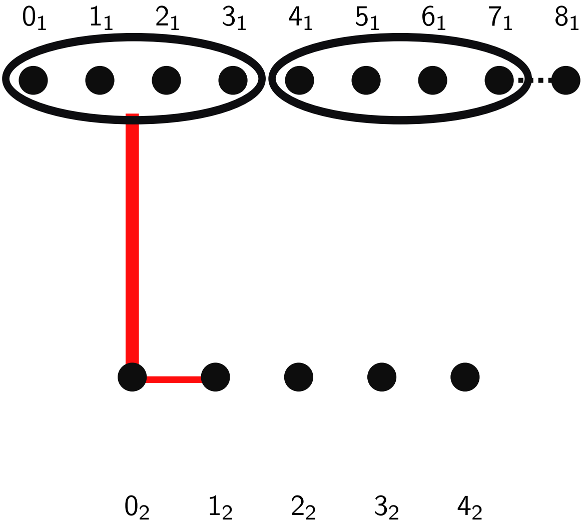

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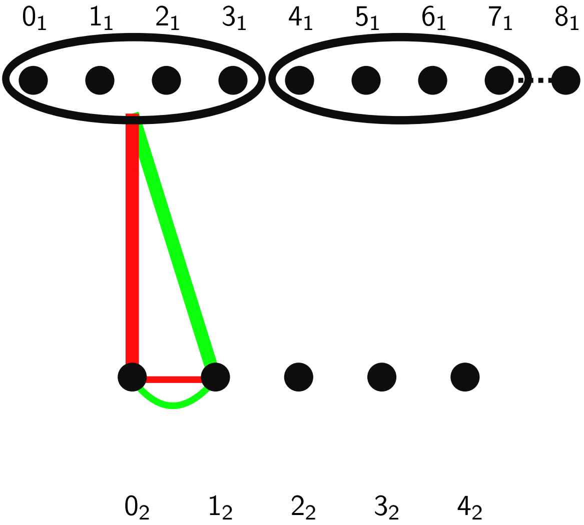

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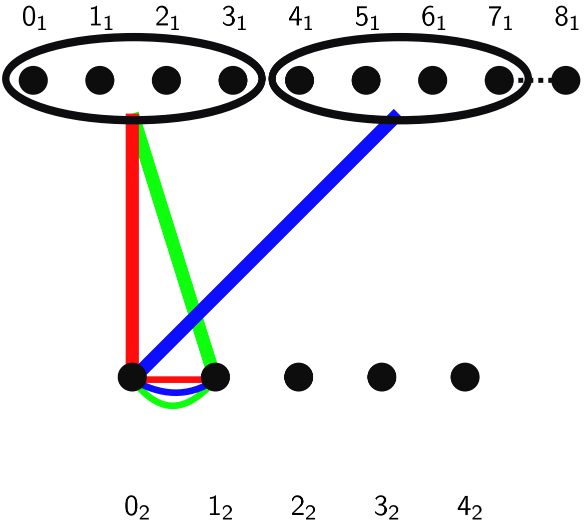

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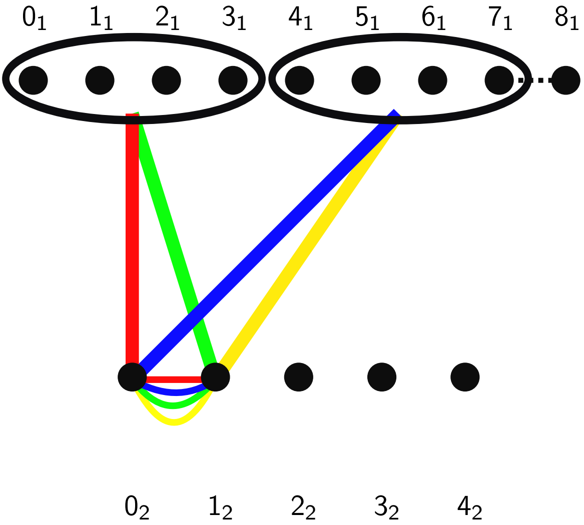

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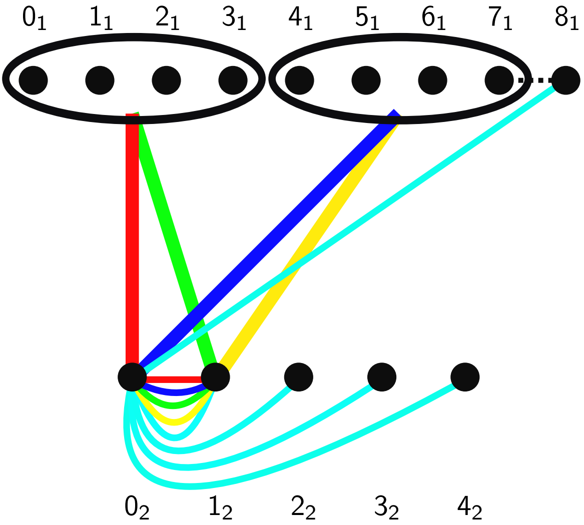

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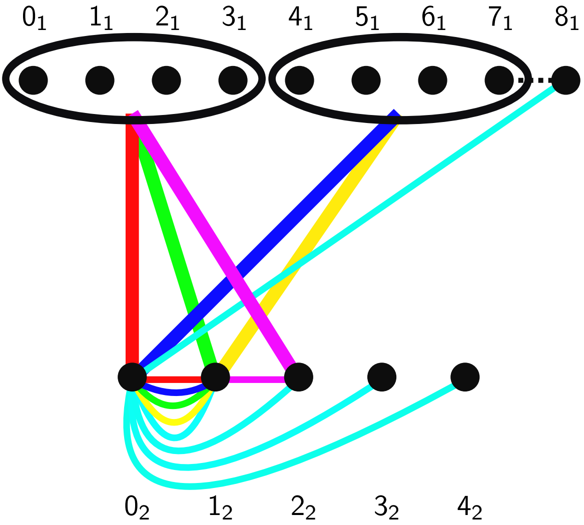

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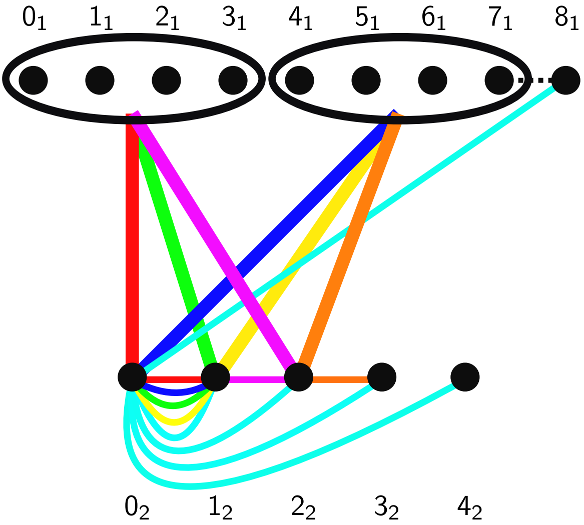

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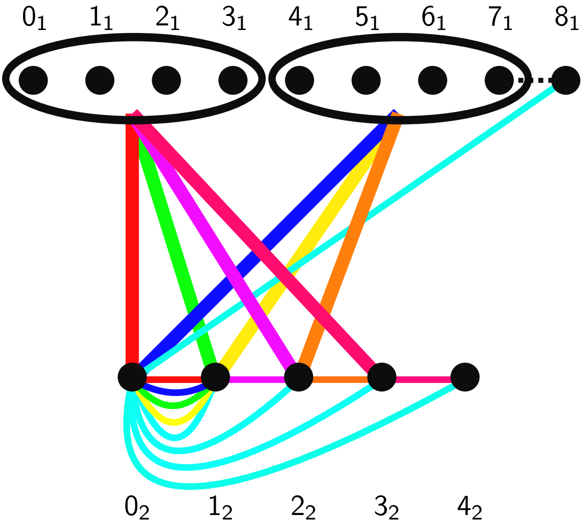

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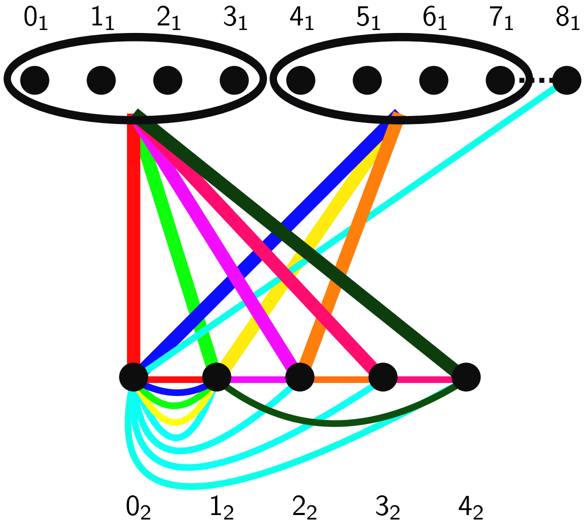

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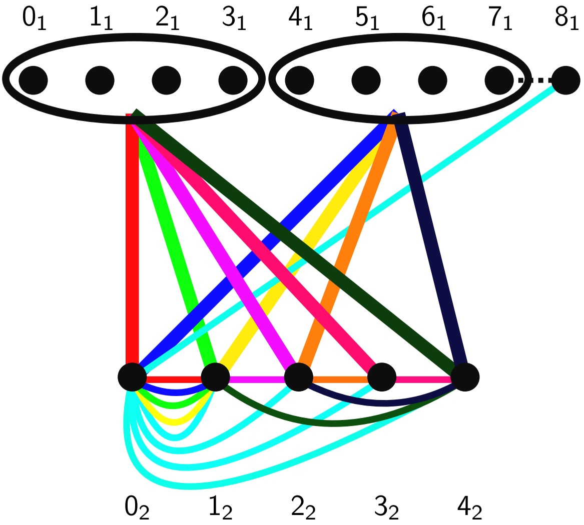

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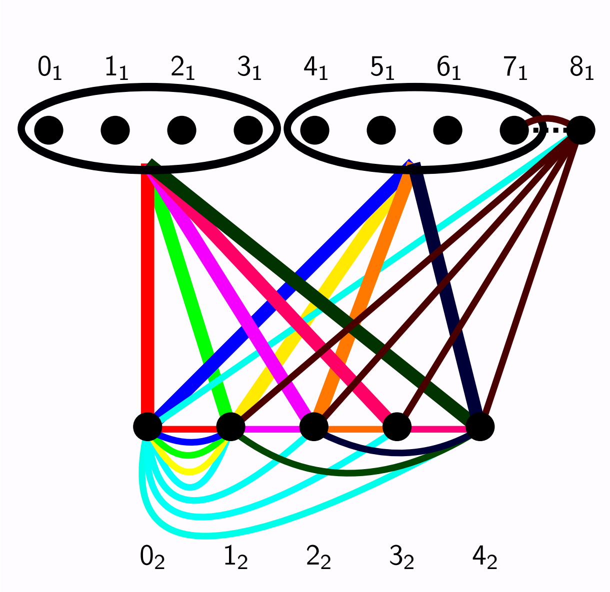

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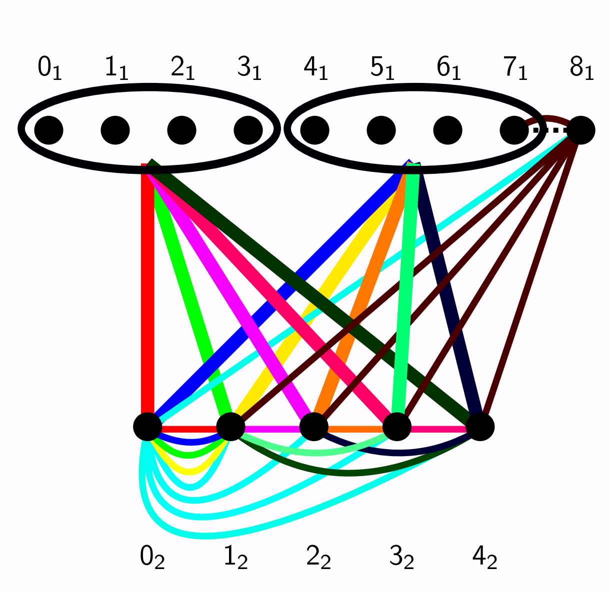

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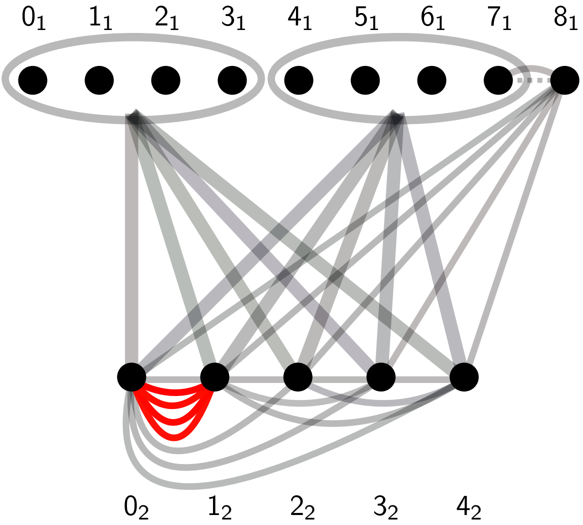

Future work

- Generalization of Theorems 1 and 2; i.e., the spectrum for packing and covering the complete graph with any S_k.
- The spectrum for packings and coverings of the complete graph K_n with all trees with up to five edges.
- Investigating different possibilities for leaves and excesses of maximal packings (minimal coverings) of the complete graph with trees with up to five edges.