Extending the parking space

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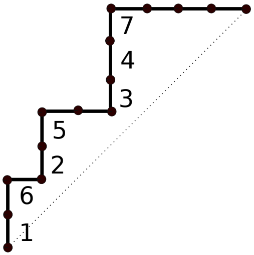
(joint with Andrew Berget)

UCSD

Parking Functions



A *parking function of size n* is a labeled Dyck path of size n:

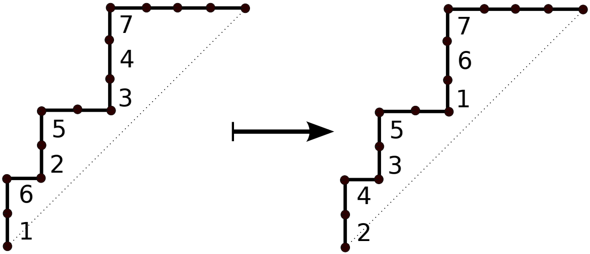
- a vertical run of size k is labeled with a subset of $[n]$ of size k,
- every letter in $[n]$ appears once as a label.

Defn: $\text{Park}_n = \{$ parking functions of size $n\}$.

Fact: $|\text{Park}_n| = (n+1)^{n-1}$.

The Parking Space

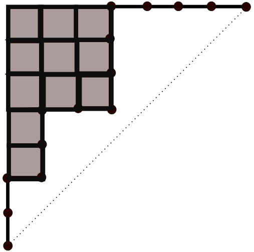
$\mathfrak{S}_n$ acts on Park_n by *label permutation*.



Q: How does Park_n decompose as an $\mathfrak{S}_n$-module?

Vertical Run Partitions

If D is a Dyck path of size n, get a *vertical run partition* $\lambda(D) \vdash n$.



$$\lambda(D) = (3, 2, 2) \vdash 7$$

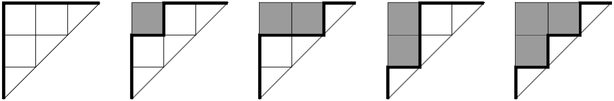
Coset Decomposition

Given $\lambda \vdash n$, let $\mathfrak{S}_\lambda$ be the *Young subgroup*.

$M^\lambda = \mathfrak{S}_n / \mathfrak{S}_\lambda =$ coset representation.

Fact: $\text{Park}_n \cong_{\mathfrak{S}_n} \bigoplus_D M^{\lambda(D)}$, where D ranges over all size n Dyck paths.

Example:



$$\text{Park}_3 \cong_{\mathfrak{S}_3} M^{(3)} \oplus 3M^{(2,1)} \oplus M^{(1,1,1)}.$$

Main Theorem

Theorem: [Berget-R] There exists an $\mathfrak{S}_{n+1}$-module V_n such that

$$\mathrm{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V_n) \cong_{\mathfrak{S}_n} \mathsf{Park}_n.$$

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Probably not.

Fact: Park_n does *not* in general extend to $\mathfrak{S}_{n+1}$ as a permutation module. Also, Park_n does *not* in general extend to $\mathfrak{S}_{n+2}$ at all.

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- The coset representation M^λ does not extend to $\mathfrak{S}_8$ for $\lambda = (3, 2, 2) \vdash 7$.
- The map

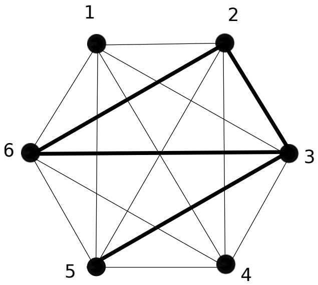
$$\mathrm{Res} : K_0(\mathfrak{S}_{n+1}) \to K_0(\mathfrak{S}_n)$$

is surjective over $\mathbb{Q}$.

Graphs

K_{n+1} = complete graph on $[n+1]$.

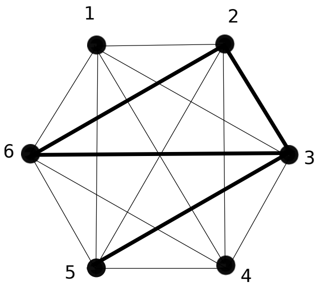
A subgraph $G \subseteq \binom{[n+1]}{2}$ is *slim* if the complement $K_{n+1} - G$ is connected.



Polynomials

To any subgraph $G \subseteq \binom{[n+1]}{2}$, we associate the polynomial

$$p(G) = \prod_{(i<j)\in G} (x_i - x_j).$$



$$p(G) = (x_2 - x_3)(x_2 - x_6)(x_3 - x_5)(x_3 - x_6)$$

Spaces

Defn: Let $V_n \subset \mathbb{C}[x_1, \ldots, x_{n+1}]$ be the subspace

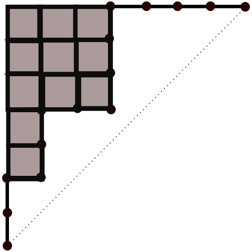
$$V_n = \mathrm{span}\{p(G) : G \subseteq K_{n+1} \text{ is slim}\}.$$

Obs: V_n is a graded $\mathfrak{S}_{n+1}$-module.

Theorem: [Berget-R] $\mathrm{Res}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n}(V_n) \cong \mathrm{Park}_n$. (Graded structure?)

Area

Defn: The *area* of a Dyck path D is the number of boxes to the northwest of D.



$$\text{area}(D) = 11$$

Graded Main Result

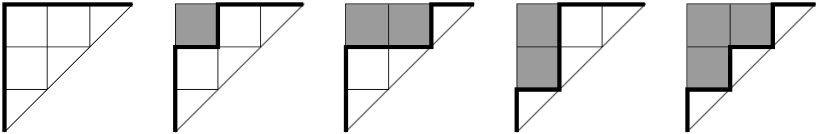
Theorem: [Berget-R] The $\mathfrak{S}_n$-isomorphism type of the degree k piece $V_n(k)$ is

$$\bigoplus_D M^{\lambda(D)},$$

where D ranges over all size n Dyck paths with area k.

Example: The graded $\mathfrak{S}_3$-character of V_3 is

$$q^0 M^{(3)} + q^1 M^{(2,1)} + 2q^2 M^{(2,1)} + q^3 M^{(1,1,1)}.$$



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Theorem: [Berget-R] Let $C = \langle (1, 2, \ldots, n+1) \rangle$ and $\zeta = e^{\frac{2\pi i}{n+1}}$. Then

$$V_n(\mathrm{top}) = V_n\left(\binom{n}{2}\right) \cong_{\mathfrak{S}_{n+1}} \mathrm{Ind}_C^{\mathfrak{S}_{n+1}}(\zeta) \otimes \mathrm{sign}.$$

Open Problems

Problem: Given a nice criterion for deciding whether an $\mathfrak{S}_n$-module M extends to $\mathfrak{S}_{n+1}$ (or $\mathfrak{S}_{n+r}$).

Problem: Determine the full graded $\mathfrak{S}_{n+1}$-structure of V_n.

Problem: For n and k fixed, what is the maximum r so that $V_n(k)$ extends to $\mathfrak{S}_{n+r}$?

- $k = 0 \Rightarrow r = \infty$
- $k = 1, n > 2 \Rightarrow r = 1$
- $k = \text{top} \Rightarrow r \geq 2$.

Thanks for listening!

A. Berget and B. Rhoades. Extending the parking space.
arXiv: 1303.5505