The University of Western Australia

Local transitivity properties of graphs and pairwise transitive designs

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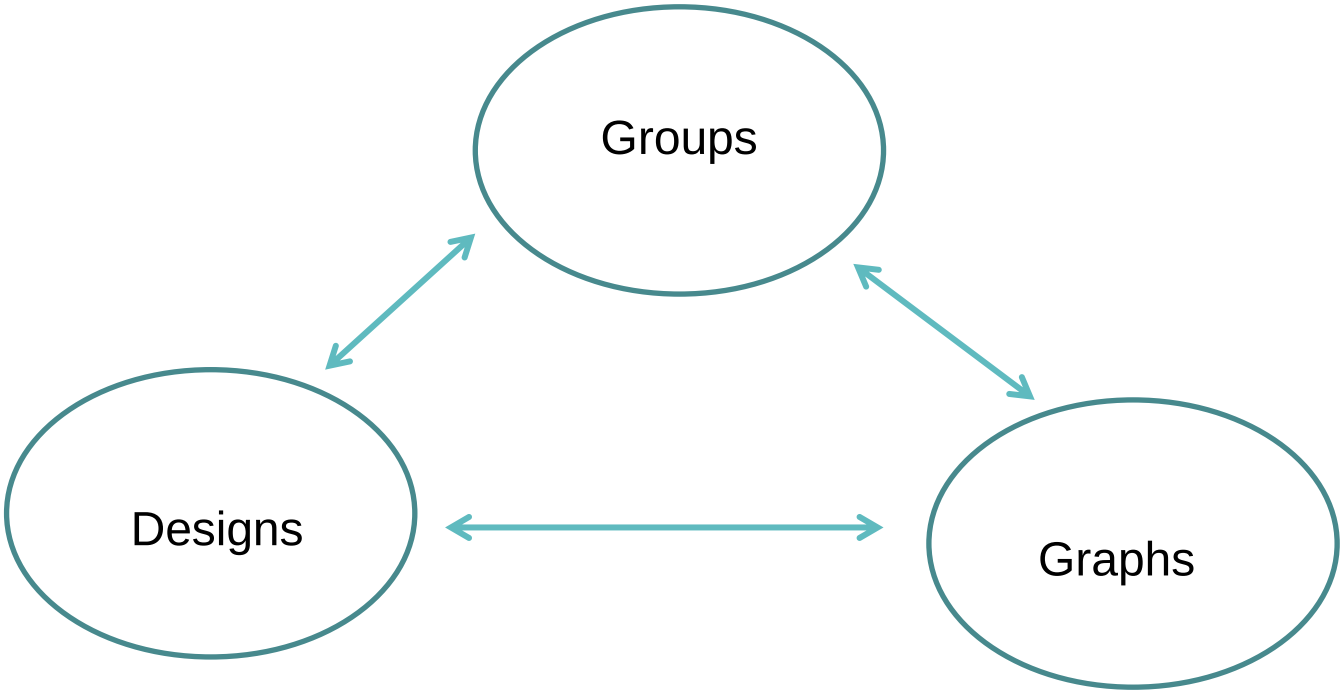
CANADAM JUNE, 2013

Interplay between different areas

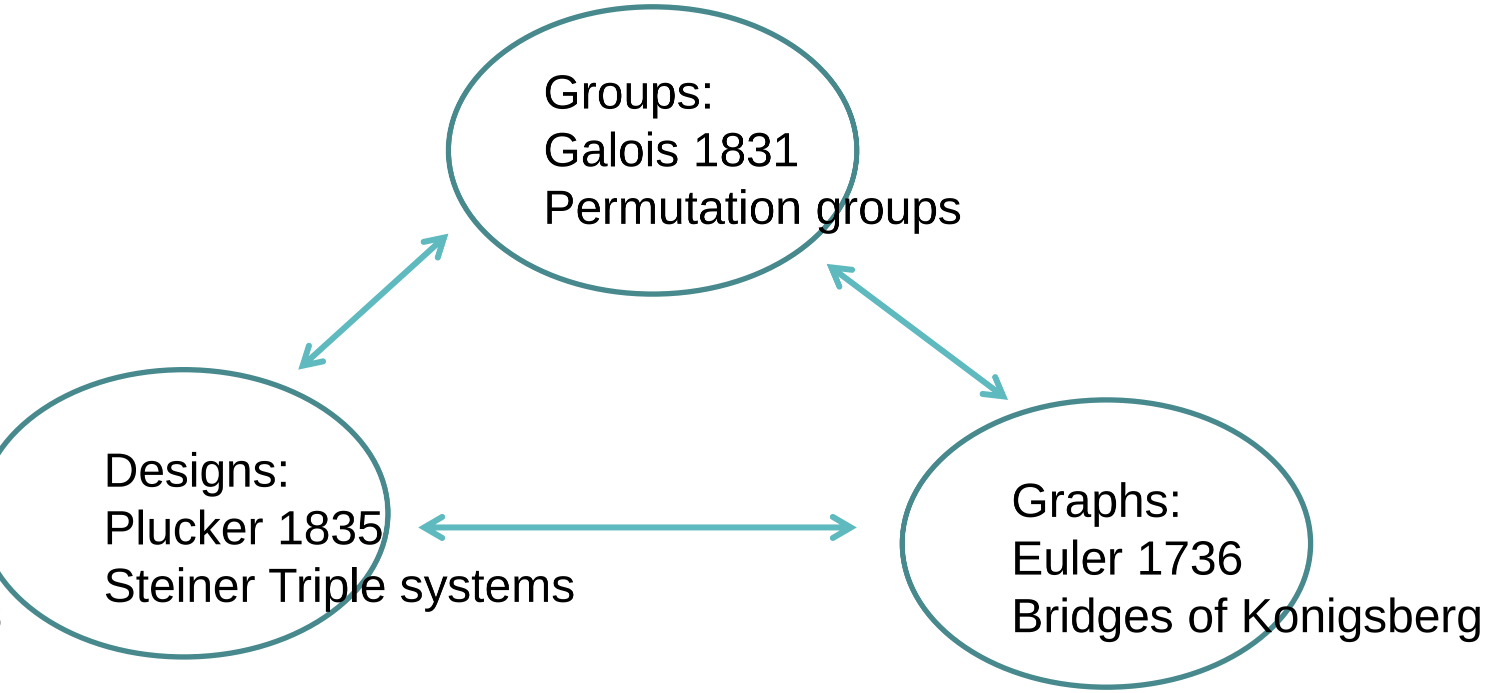


History: separate beginnings

[Leonard Euler](#) 1756

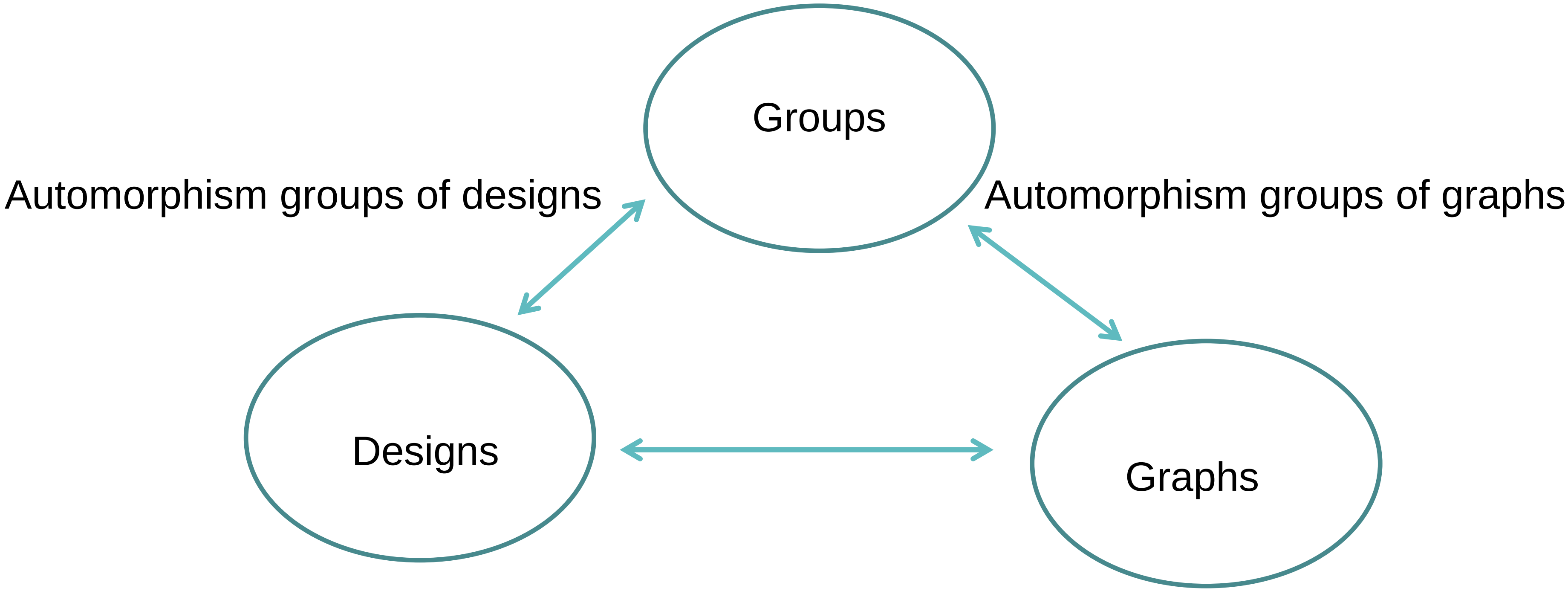
[Evariste Galois](#) 1811-1832

[Julius Plucker](#) 1856



c.1935
Statistical
analysis of
experiments

These days: many interactions



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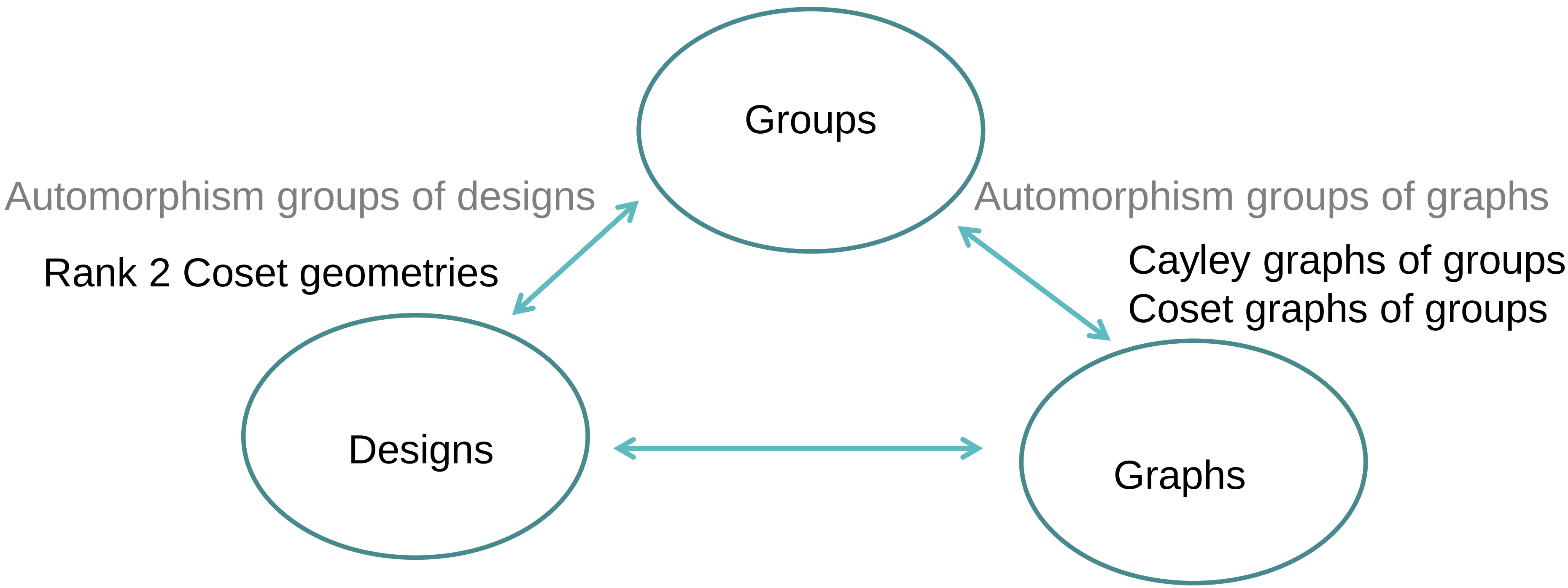
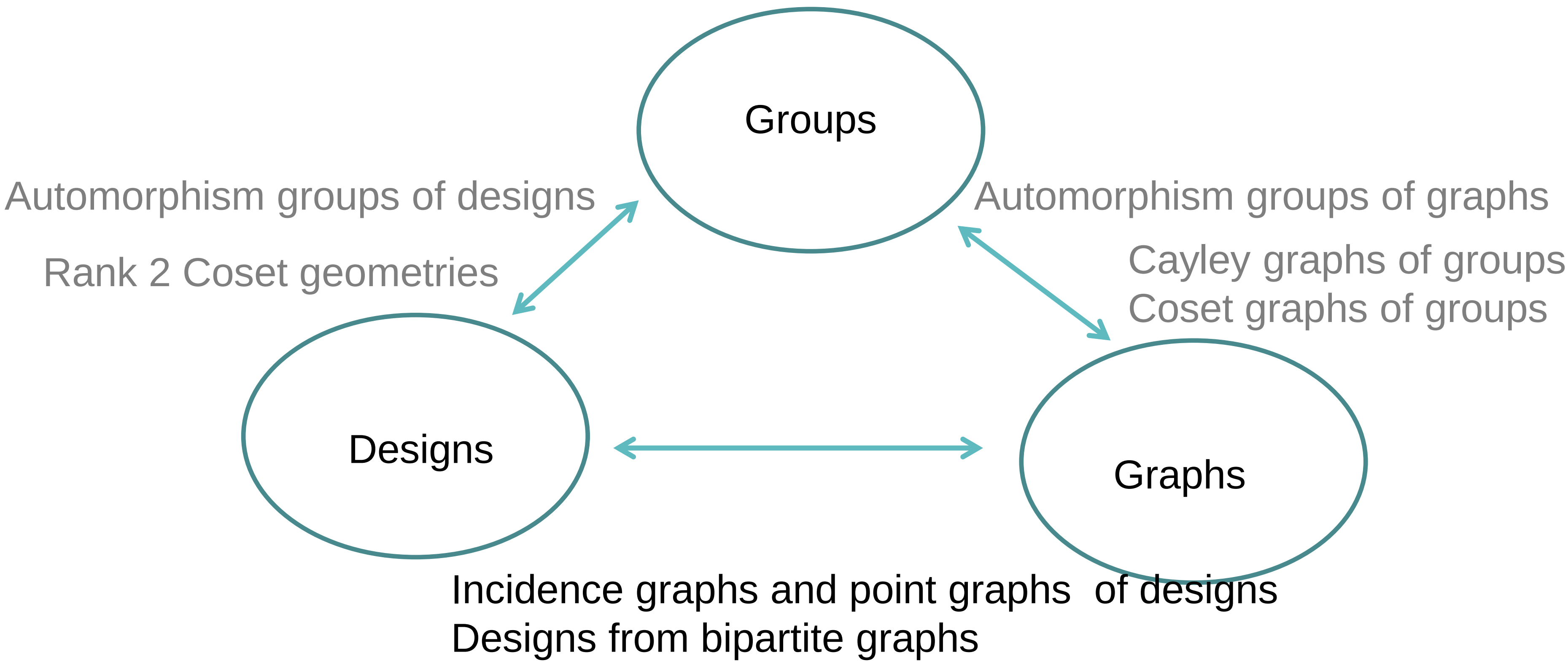
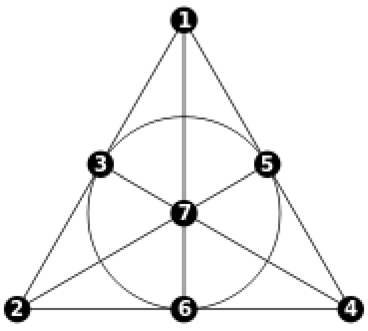


These days: many interactions

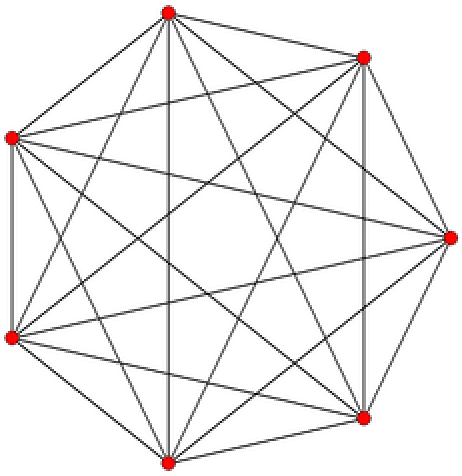


By a design we mean . . .



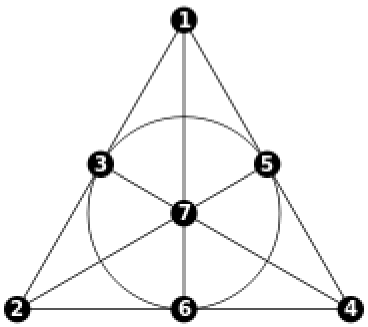
Fano plane. Courtesy: Gunther and Lambian

- Design D = (P, B, I) P set of points B set of blocks
  - I incidence relation I subset of P x B
  - Sometimes special conditions: e.g. D is a 2-design if each 2-subset of points incident with constant number of blocks
- Point graph of D: vertex set P – join if “collinear”
  - Point graph of 2-designs are complete graphs
- Incidence graph Inc(D): vertices are points and blocks, joined if incident



Point graph of Fano plane. Courtesy: Tom Ruen

By a design we mean . . .



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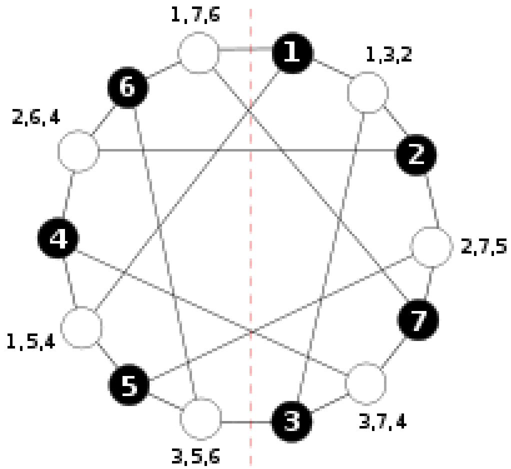
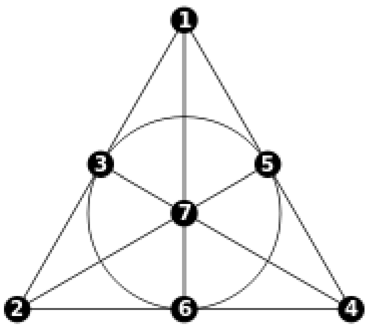
- Incidence graph Inc(D): vertices are points and blocks, joined if incident
  - Note Inc(D) is a bipartite graph with "biparts" P, B



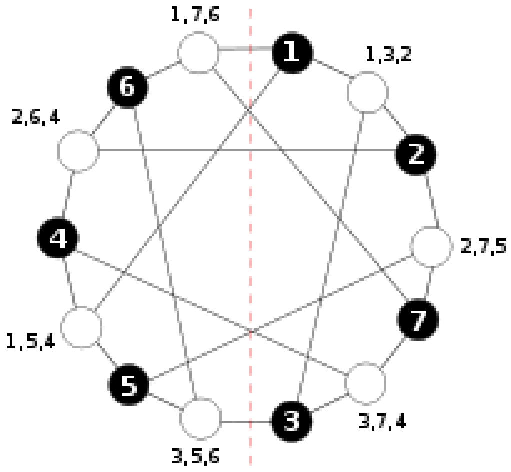
Heawood graph – the Incidence graph of Fano plane. Courtesy: Tremlin

Reverse construction:
From a bipartite graph X with biparts W, B . . .



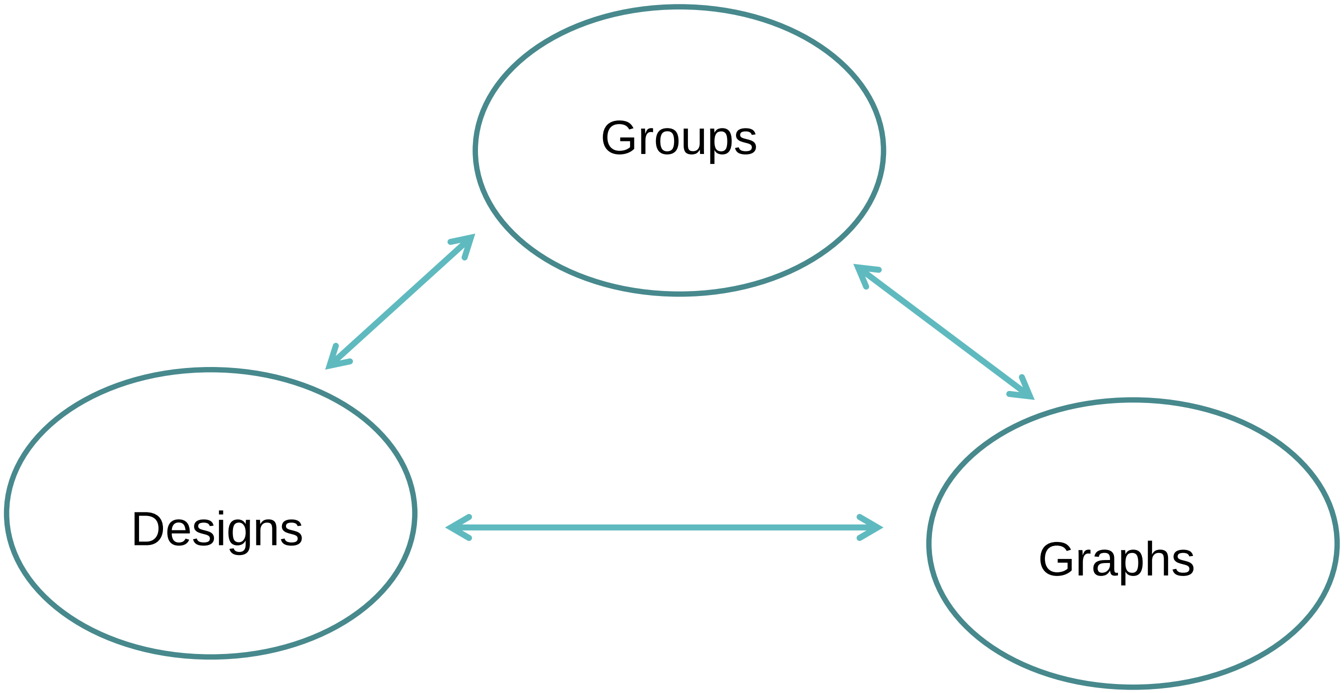
Fano plane. Courtesy: Gunther and Lambian

- Incidence Design   IncDesign(X) = (W, B, I)   W set of points   B set of blocks
  - I incidence relation { (x,y) where W-vertex x joined to B-vertex y }
  - For the Heawood graph X we get IncDesign(X) = Fano plane
- For a design D=(P,B,I): IncDesign(Inc(D)) is either D or its dual design $D^c = (B,P,I^c)$

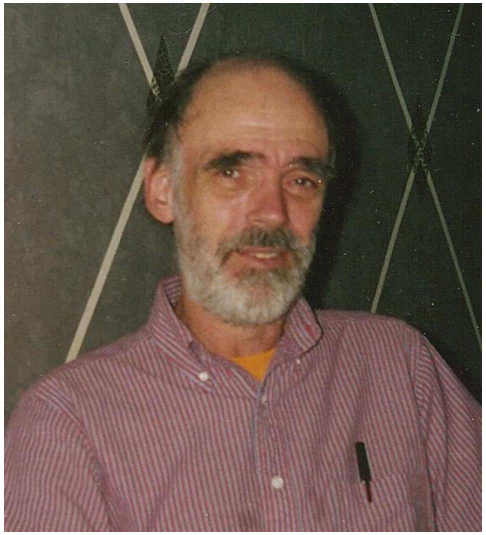


Heawood graph – the Incidence graph of Fano plane. Courtesy: Tremlin

Story of the lecture is horizontal link – for “very symmetrical” graphs and designs – so groups involved also

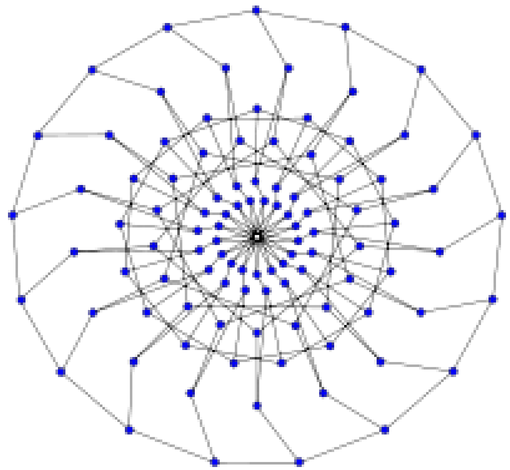


1967 D G Higman's "intersection matrices paper"



- ↘ Studied transitive permutation group G on set V
- ↘ Realised importance of G-action on ordered point-pairs V x V
  - G-orbits in V x V called orbitals
  - Interpreted as arc set of digraph, and if symmetric, of an undirected graph called orbital graph
- ↘ Initiates investigation of distance transitive graphs (without naming them)
- ↘ Suppose G has exactly r orbitals – r called the rank of G
- ↘ Imagine a connected orbital graph where for each distance j, the ordered point-pairs at distance j form just one orbital – no splitting. Then diameter of graph is r-1 , maximum possible given r
- ↘ DGH says G has maximal diameter if there exists an orbital graph like this

1971 Norman Biggs



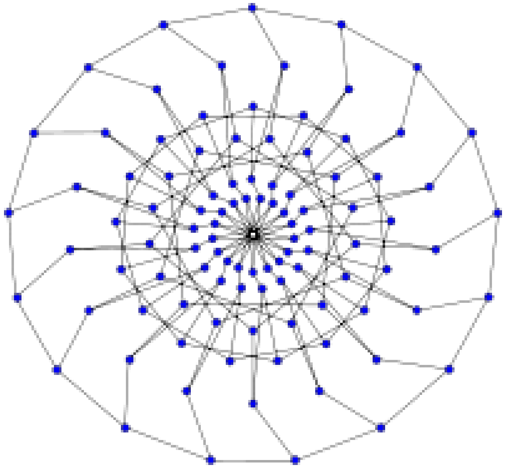
Biggs-Smith graph – 102 vertices Courtesy: Stolee

- Focus turns to the graphs: called them distance transitive graphs (DTGs)
- Biggs and Smith: exactly 12 finite valency three DTGs

- Suppose X DTG with diameter d and automorphism group G
  - d=1 complete graphs K_n
  - d=2 complete multipartite graphs $K_{n[m]}$, or “primitive rank 3 graphs”
  - “primitive rank 3 graph”: G vertex-primitive and rank 3 – all such DTGs known [use of FSGC is common method in these investigations]

- d > 2 D. H. Smith: gave two constructions to reduce a given DTG to a smaller DTG - bipartite halves, antipodal quotients. after at most 3 applications obtain a vertex-primitive DTG

Primitive DTGs X   G=Aut(X)



- G vertex-primitive: no G-invariant vertex partitions

Biggs-Smith graph – 102 vertices Courtesy: Stolee

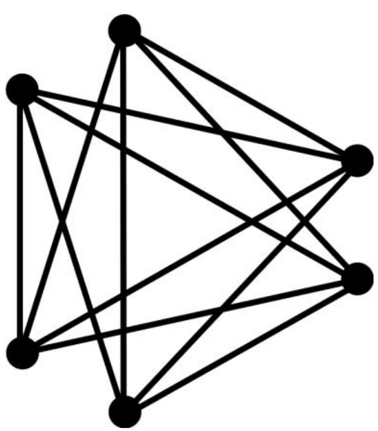
- Powerful analytical tools available: O'Nan-Scott Theorem
  - can(with hard work) reduce to cases where we can apply FSGC and representation theory

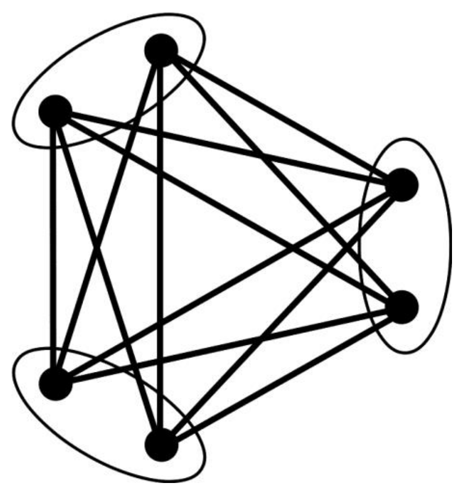
- 1987 CEP Saxl Yokoyama: G almost simple, or G affine, or X known (Hamming graph or complement)

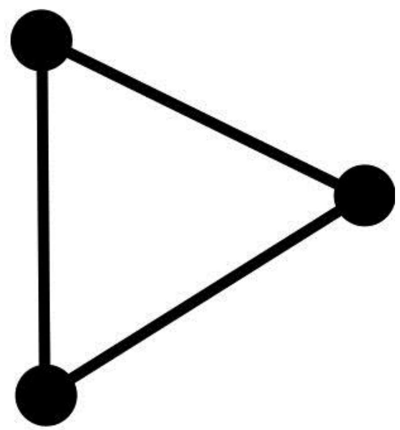
- 2013? "just a few almost simple cases to be resolved" Arjeh Cohen's 2001 web survey: http://www.win.tue.nl/~amc/oz/dtg/survey.html

Quotient Construction: applicable to other graph families

- Partition vertices
- Merge vertices in each part to a single vertex
- Like viewing from afar with a telescope so fine details disappear revealing the essential features.
- Trick: do this while preserving the symmetry



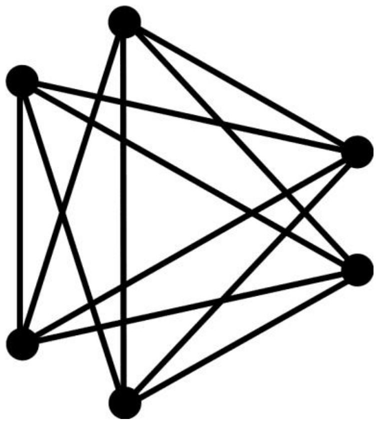


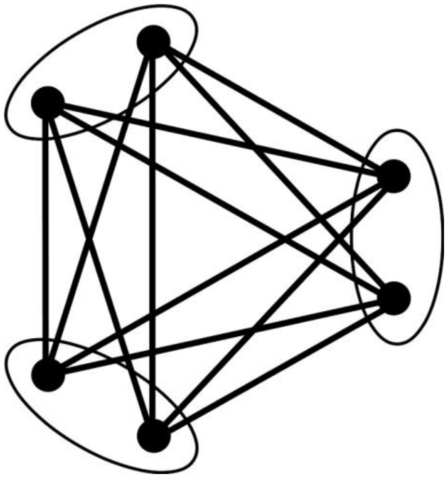


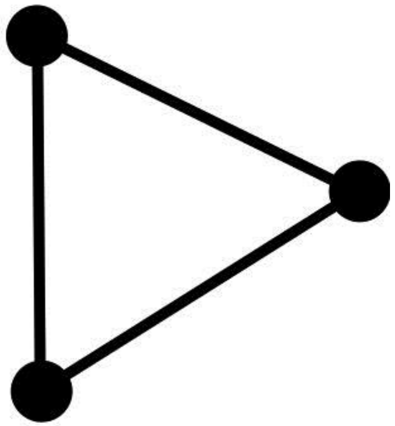
Graph Images. Courtesy: Geoffrey Pearce

Quotient Construction: applicable to other graph families

- Trick: do this while preserving the symmetry
- Quotients modulo G-invariant partitions of graph X admit action of G – but action may not have all desirable properties
- Special G-invariant partitions: G-Normal partitions often good. Orbit set of normal subgroup N of G. Produce G-normal quotient X_N







Graph Images. Courtesy: Geoffrey Pearce

Alice Devillers
Michael Giudici
Cai Heng Li

Locally s-distance transitive graphs X relative to group G

- More general than DTGs
- 1. “s” at most diameter d of X -- “locally”
- 2. Require: from each vertex x, and for all j at most s, all vertices at distance j from x form single G_x-orbit

- Reduction to vertex-primitive case impossible instead …
- Normal quotients X_N either still locally s-distance transitive or some degeneracies occur.

- Degenerate quotients:
- N transitive $X_N = K_1$
- X bipartite and N-orbits are the bipartition $X_N = K_2$
- X bipartite and N transitive on only one bipart X_N is a star $K_{1,r}$

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Locally s-distance transitive graphs X relative to group G

- Degenerate quotients:
- N transitive — $X_N = K_1$
- X bipartite and N-orbits are the bipartition — $X_N = K_2$
- X bipartite and N transitive on only one bipart — X_N is a star $K_{1,r}$

- Other Milder degeneracies: diameter of quotient X_N may be less than s

- Theorem: Either X_N is degenerate, or G acts locally s'-distance transitively on X_N where s' = min{ s, diam(X_N) }

- Consequence: study basic locally s-distance transitive graphs X – non-degenerate, but all G-normal quotients degenerate.

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Basic Locally s-distance transitive graphs X relative to group G

- Basic graphs – X is non-degenerate, but all G-normal quotients degenerate
- Admit actions of group G related to quasiprimitive groups
- Quasiprimitive groups: larger class than primitive groups & have similar tools for their study (an "O'Nan-Scott Theorem" (CEP 1993) – links to representation theory and use of FSGC)
- Because of this approach we found an interesting link with designs

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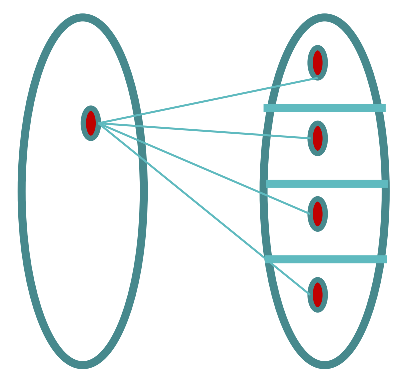
We studied locally s-distance transitive graphs X with a star-normal quotient $K_{1,r}$ relative to group G

- Star normal quotient X_N – X bipartite, N transitive on left side, r orbits on right side

- How large can s be?
- Could prove s at most 4 - wondered if s could be equal to 4
- Each vertex on left joined to exactly r vertices on right, one in each N-orbit

- X is bipartite graph so consider D = IncDesign(X)
-
- Points P on left, Blocks B on right
- Each N-orbit in B is a “parallel class” of blocks

- D is resolvable



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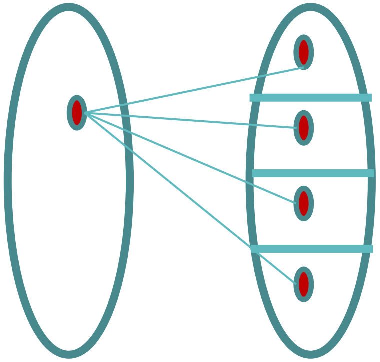
Properties of D = IncDesign(X) if G is locally 4-distance trans

G is transitive on P, B and on “all kinds of ordered pairs” from P or B

- Collinear point-pairs [incident with common block]
- Non-collinear point-pairs
- Incident point-block pairs [flags]
- Non-incident point-block pairs [anti-flags]
- Intersecting block pairs
- Non-intersecting block pairs   [some may be empty]



- Call such designs D pairwise-transitive

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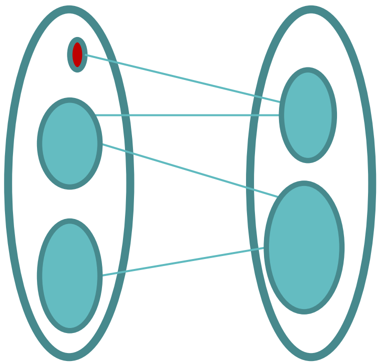
Interesting converse: if D is G-pairwise-transitive then G is locally 4-distance transitive on X=Inc(D)

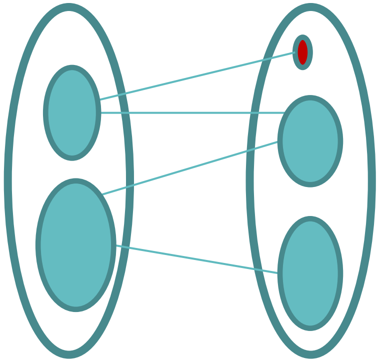
G is transitive on P, B and …..

G_p transitive on vertices at
Distance 1 [flags]
Distance 2 [collinear points]
Distance 3 [antiflags]
Distance 4 [non-collinear points]

p



b



So all pairwise-transitive designs interesting – all give locally 4-DTGs – but not all give “star-like” DTGs [have normal subgroup N with $X_N = K_{1,r}$]

Plenty of examples of pairwise transitive designs

- Classical Examples:
- P=V(d,q), B=affine hyperplanes, G = AGL(V)
- P=PG(d,q), G=PGL(d+1,q) B = projective hyperplanes, or hyperplane complements, of projective lines
-
- Sporadic examples: e.g. Mathieu-Witt design (Steiner system) 3-(22,6,1) [each triple in exactly one block] G = M_{22}
- "Little bit different examples":
- P=V(d,q), Choose a line u through origin
- B=affine hyperplanes not containing line parallel to u.
- G = $AGL(V)_{[u]}$

Alice Devillers

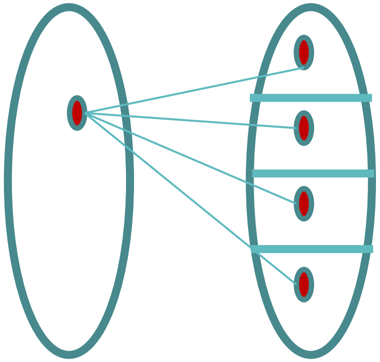
What is known about pairwise transitive designs?

What existing studies of transitivity on:

- Collinear point-pairs [incident with common block]
- Non-collinear point-pairs
- Incident point-block pairs [flags]
- Non-incident point-block pairs [anti-flags]
- Intersecting block pairs
- Non-intersecting block pairs   [some may be empty]

Property is “self-dual” – dual design also pairwise transitive

Some of these properties studied before, but not all together



What some of the conditions mean:

- Transitive on collinear (ordered) point-pairs:
  - if all pairs collinear then G 2-transitive on P and all point-pairs incident with same number x of blocks D is a 2-design
  - if not all pairs collinear then G has rank 3 on P (3 orbits on P x P)
- Similarly G is 2-transitive or rank 3 on B

- Case: Point and block 2-transitive: D is a symmetric 2-design [|P| = |B|]

- In fact for D symmetric 2-design:

  G 2-transitive on P if and only if G is pairwise transitive on D

Pairwise transitive symmetric 2-designs

- 1985 Kantor classified them all [using FSGC]
-
- Examples:
  - Trivial designs all (v-1)-subsets of a v-set
  - Points-Hyperplanes of PG(d,q)
  - Hadamard 2-(11,5,2) biplane G=PSL(2,11)
  - Higman-Sims 2-(176,50,14) design G = HS
  - Symplectic designs: point set V = V(2m,2) with symplectic form (nondegenerate, alternating)
  - Block set another copy of V. $G=[2^{2m}].Sp(2m,2)$
- Together with complementary design D^c [take complements of blocks]

2-transitive on points P

- all point-pairs collinear - all incident with same number x of blocks
- If x=1 then D is called a linear space
- 1985 Kantor classified point 2-transitive linear spaces [using FSGC]
  - Not necessarily all are pairwise transitive

- Linear spaces + transitivity on flags (incident point-block pairs)
- 1990 Buekenhout, Delandtsheer, Doyen, Kleidman, Liebeck: Classification announced of flag-transitive linear spaces
  - Up to an open “1-dimensional affine case”
  - for which Kantor (1993) believes classification “completely hopeless”
  - Last paper containing proofs: 2002

Previously studied pairwise transitivities

- Transitivity on anti- flags (non-incident point-block pairs)
- 1984 Delandtsheer. Classified anti-flag transitive linear spaces
- 1979 Cameron and Kantor. Classified groups of semilinear transformations anti-flag transitive on point-hyperplane projective designs
- Don’t know any studies for transitivity on intersecting or non-intersecting block pairs

Where are we at: AliceD and Cheryl

- Classified all pairwise transitive 2-designs: includes
  - 2-transitive symmetric designs classified by Kantor
  - other classical examples mentioned before (AG, points and lines of PG)
  - three sporadic examples [involving $G=M_{11}$, M_{21}, M_{22}]
- This leaves: D not a 2-design, so there are non-collinear point pairs.
  - Since dual design also pairwise transitive, can assume that dual not a 2-design either
  - So there are also non-intersecting blocks
  - G is a rank 3 group on points and on blocks
- All primitive rank 3 groups are known [FSGC] so we think we can use this classification to find all point-primitive pairwise transitive designs.
- Imprimitive rank 3 groups not all known but … since also G rank 3 on blocks, maybe ….
- Also would like to know which are star-like

Thank you

Photo. Courtesy: Joan Costa joancostaphoto.com