WELL-COVERED QUADRANGULATIONS

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Let $\alpha(G)$ denote the independence number of G;

i.e., the size of a largest independent set of vertices.

Independent set problems are hard!!!

And no wonder!

Theorem (Karp 1972): Determining $\alpha(G)$ is NP-complete.

And the problem remains NP-complete, even if:

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1. G is triangle-free

   or

2. G is cubic planar

   or

3. G is $K_{1,4}$-free.

OK! I'm happy to *approximate* $\alpha(G)$

(in polynomial time)!!!

HAH!!!!

You CANNOT approximate closer than

$$1.36\alpha(G)$$

(in polynomial time)

Unless

$$P = NP$$

So when is finding $\alpha(G)$ $easy$???

It is trivially *easy* (i.e.,*polynomial*) to find $\alpha(G)$ if

every maximAL independent set is maximUM.

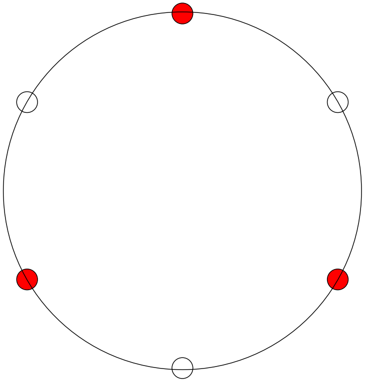
Just start with any vertex and build your independent set in a *greedy* manner!

Graphs with this property are called well-covered.

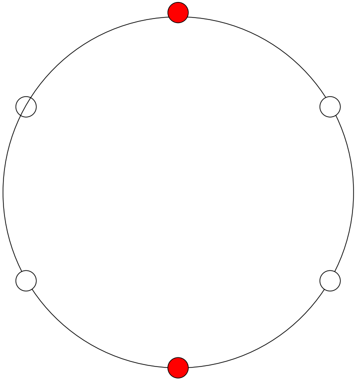
Examples: C_3, C_4, C_5, C_7

But NOT C_6!

C_6



C_6



Great!!!!!

.....but.....

When is a graph well-covered?

Can these graphs be recognized in polynomial time???

Well, given a non-well-covered graph G, I hand you two maximal independent sets of differing cardinalities. You can check their maximality in polynomial time.

So recognizing a non-well-covered graph is in co-NP.

Actually, the problem is known to be co-NP-complete!

(Chvátal-Slater (1993); Sankaranarayana-Stewart (1992))

And it remains co-NP-complete, even for

circulant graphs

.

(Brown and Hoshino, 2011)

But the complexity of the recognition problem for graphs that *are* well-covered remains

UNKNOWN!!!

Finbow, Hartnell and Nowakowski (1993) characterized well-covered graphs having

girth at least 5

and their characterization leads to a

polynomial recognition algorithm

So it remains to focus on

girth 3 and 4

PROBLEM (2011):

Characterize well-covered planar quadrangulations

Lemma: A planar quadrangulation

(a) contains no triangles

and

(b) is bipartite.

Part (b) follows from part (a) and induction.

Ravindra's Theorem: A bipartite well-covered graph G contains a perfect matching and for every perfect matching M in G and for every edge e in M, $G[N(x) \cup N(y)]$ is a complete bipartite graph.

So in particular, a bipartite well-covered graph must be balanced.

Let us denote by WCQ, the set of all well-covered quadrangulations of the plane.

Theorem: Suppose $G \in WCQ$, M is a perfect matching in G and $e = xy$ is an edge in M. Then either $G = C_4$ or else exactly one of x and y has degree 2 in G.

(Hence, if $G \neq C_4$, half the vertices of G have degree 2 and the rest have degree at least 3.)

Now define a second set of quadrangulations of the plane, denoted by WCQ', as follows:

Def.: A quadrangulation Q' belongs to WCQ' if there is a set of vertex-disjoint 4-cycles, $C_1, C_2, \ldots, C_k$ in the plane

(we call these *basic* 4-cycles)

such that $V(Q') = V(C_1) \cup \cdots \cup V(C_k)$ and each pair of basic 4-cycles are joined according to the following recipe:

Either the pair are joined by no edges

or

they are joined precisely as shown in Figure 1 below:

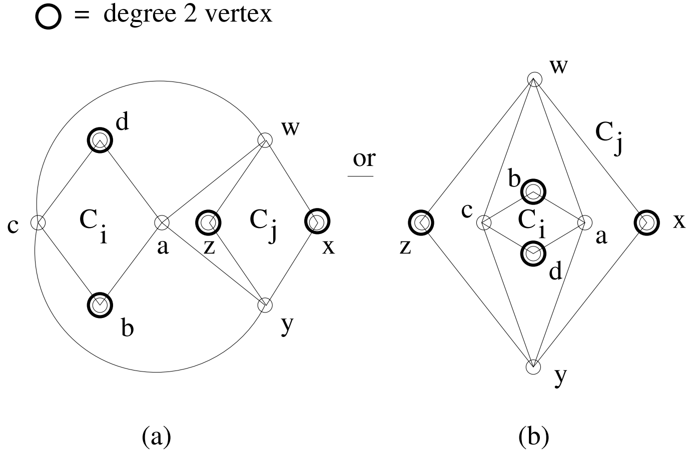
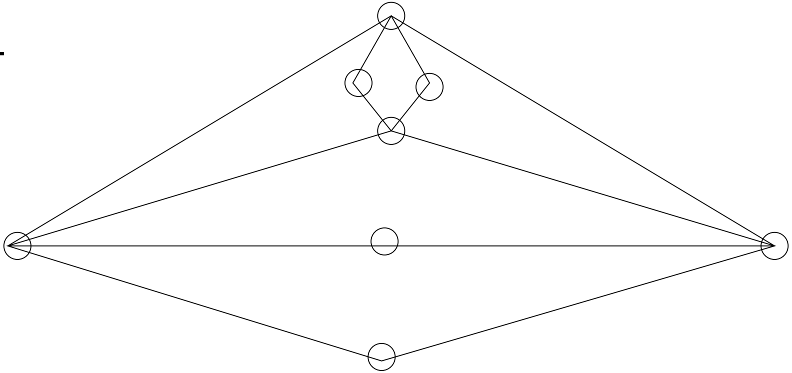


Figure 1

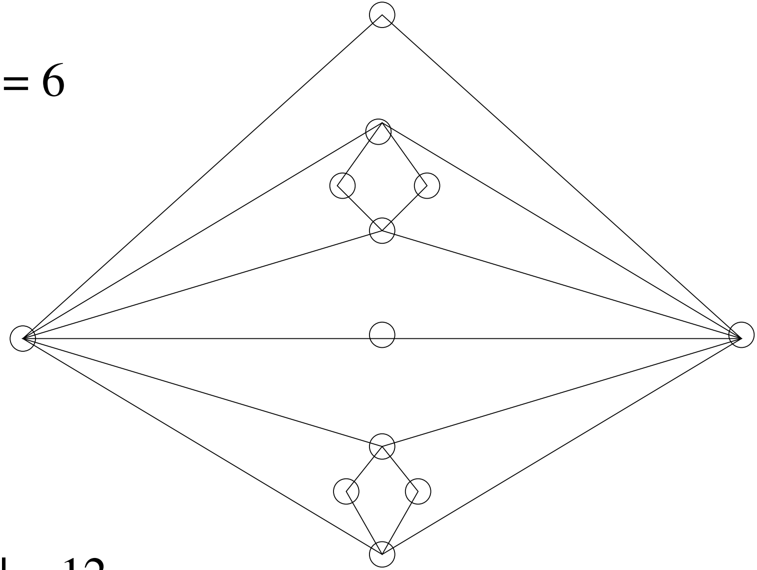
Here are some examples of graphs belonging to WCQ':

$\alpha(G) = 4$



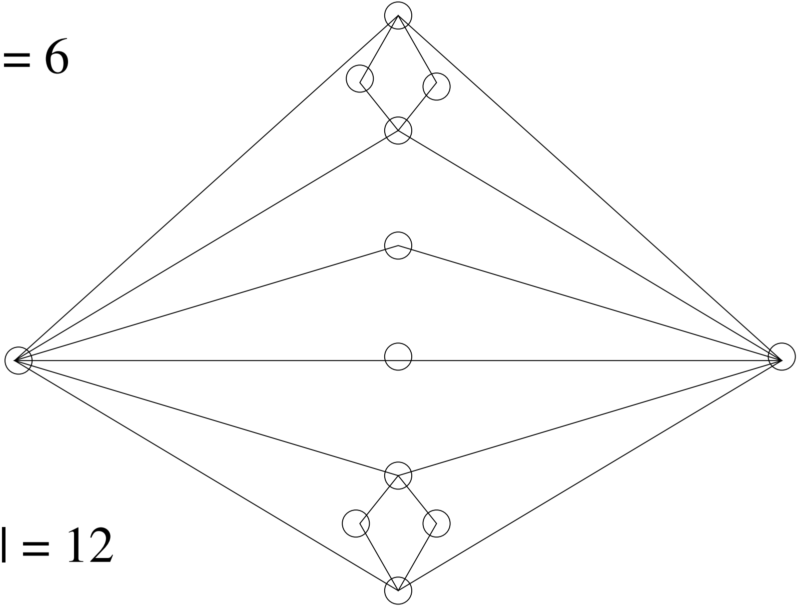
$|V(G)| = 8$

$\alpha(G) = 6$



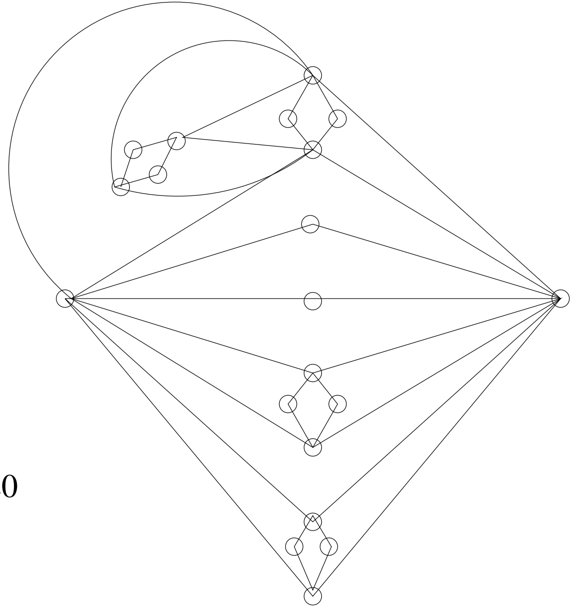
$|V(G)| = 12$

$\alpha(G) = 6$



$|V(G)| = 12$

$\alpha(G) = 10$



$|V(G)| = 20$

Main Theorem: $WCQ = WCQ'$.

Proof: $WCQ \subseteq WCQ'$: Argument uses Ravindra's Theorem repeatedly.

$WCQ' \subseteq WCQ$: If $G = C_4$, this is clear.

If $G \neq C_4$, we argue that any maximum independent set I in G must contain *precisely two* vertices from each basic 4-cycle.

Recognition of graphs in WCQ is clearly *polynomial*.

1. Find a perfect matching M.

   (If none exists, $G \notin WCQ$.)

2. By Ravindra's theorem, if $G \neq C_4$, each edge of M must have a vertex of degree 2 in G.

Use M and Ravindra's theorem via the method used in the Main Theorem to build a set of basic 4-cycles.

Note that, if $G \neq C_4$, each basic 4-cycle contains *exactly two* vertices of degree 2. If the process fails, G is not well-covered.

3. Now test every pair of basic 4-cycles to see that either they are joined by no edge or they are joined precisely as in Figure 1 above.

4. If each pair are so joined, G is in WCQ.

If there is a pair that are not so joined, G is *not* in WCQ.

— THE END —

PROBLEM (1988):

Characterize well-covered planar *triangulations*

This has proved *much harder* than quadrangulations!!!

A ROADMAP:

1. 5-connected:

There are none!

(Finbow, Hartnell, Nowakowski +MDP 2004)

2. 4-connected:

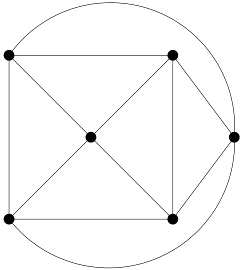
There are precisely 4 !

This was done in *two steps:*

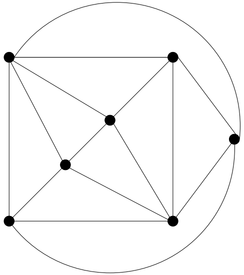
(a) If a 4-connected well-covered triangulation contains two adjacent vertices of degree 4, then there are precisely four such graphs.

(FHNP 2009)

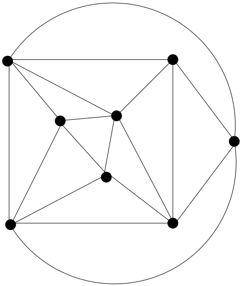
R_6



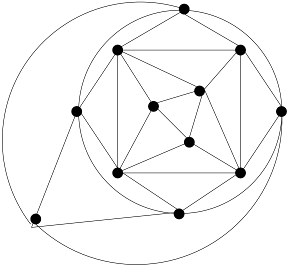
R_7



R_8



R_{12}

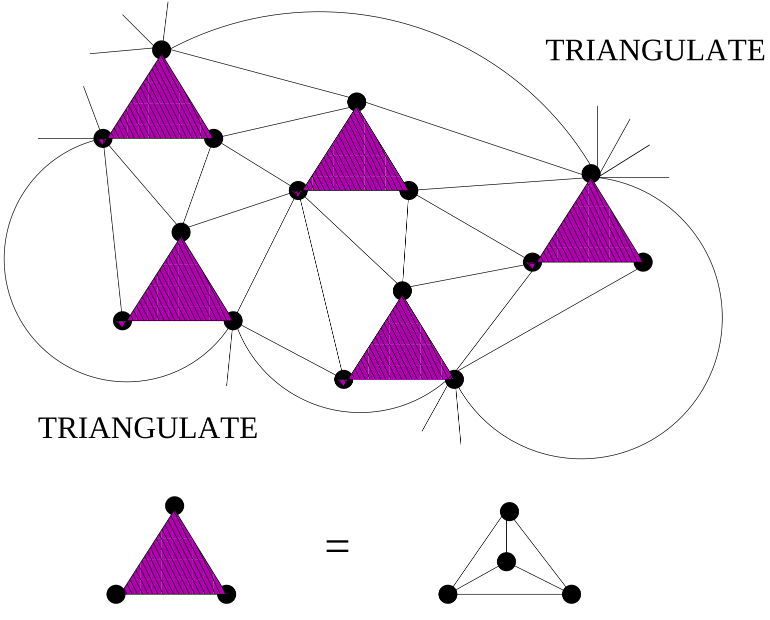


(b) Every 4-connected well-covered triangulation must contain two adjacent vertices of degree 4.

(FHNP 2010)

3. What about 3-connected triangulations??????

Here is an infinite family:


TRIANGULATE
TRIANGULATE
=

The family of such graphs is called the K_4-family and is denoted by $\mathcal{K}$.

BUT.....these are NOT ALL!!!

Flash!!!

The family is now characterized and is polynomially recognizable (FHNP 2012).

The paper is some 40 pages long (!), so we will give just an outline:

Lemma: If G is well-covered and v is a vertex in G, then $G - N[v]$ is well-covered.

Applying this lemma repeatedly, it is easy to see that

Lemma: If G is well-covered and $I = \{v_1, \ldots, v_k\}$ is an independent set in G, then $G - N[I] = G - (\cup_{i=1}^{k} N[v_i])$ is also well-covered.

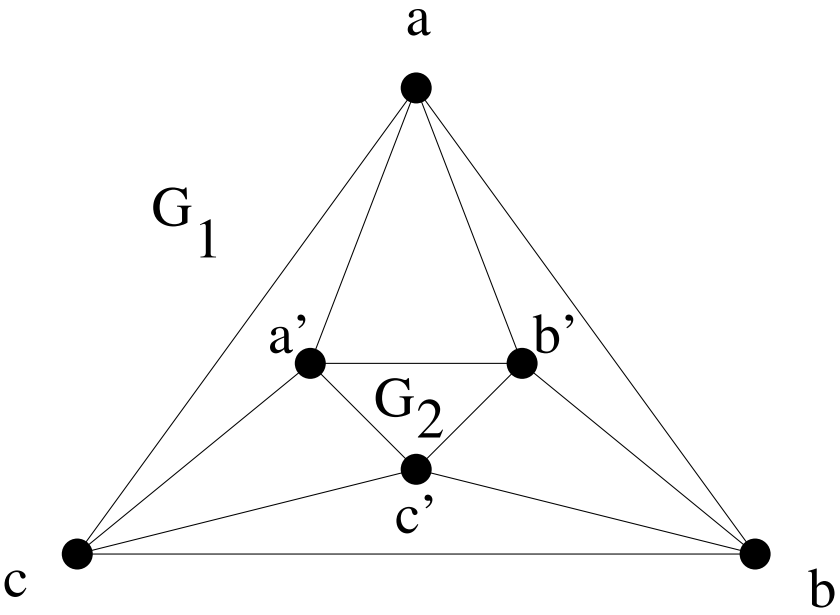
We often use the preceding lemma to show that a certain graph is *not* well-covered, by strategically finding an independent I in G such that $G - N[I]$ is *not* well-covered and therefore the parent graph is *not* well-covered.

BUT it can be very difficult to find just the right independent set I here!

Next we need a new concept called O-join.

Suppose that G_1 and G_2 are both 3-connected planar triangulations and that G_1 contains a triangular face $abca$ and G_2, a triangular face $a'b'c'a'$. Embed G_1 so that $abca$ is an *interior* face and embed G_2' so that $a'b'c'a'$ bounds the *infinite* face.

Let $G_1 \bigcirc G_2$ denote the graph obtained by embedding G_2 into the interior of face $abca$ of G_1 and adding the six edges shown in the following figure.


a
G1
a'
b'
G2
c'
c
b

Then $G_1 \bigcirc G_2$ is called an O-join of G_1 and G_2 at the faces $abca$ and $a'b'c'a'$.

(The "O" in "O-join" stands for "octahedral".)

(Note that given two triangles labeled as above, there are *six* possible O-joins at these triangles.)

Theorem: If G_1 and G_2 are each 3-connected planar well-covered triangulations, then any O-join $G_1 \bigcirc G_2$ is also a 3-connected planar well-covered triangulation.

The converse of this theorem is

MUCH MORE DIFFICULT!!

In fact, most of this long paper is devoted to showing that:

if G is a 3-connected planar well-covered triangulation and G is not one of ten exceptional graphs, then G must be constructed from smaller members of the family via a succession of O-joins.

Def.: Let G be a well-covered triangulation and $abca$, a face of G. Then $abca$ is called a YES-face if $G-a-b, G-a-c$ and $G-b-c$ is also well-covered.

A triangular face which is not a YES-face is called a NO-face.

Lemma: Suppose G_1 and G_2 are planar triangulations O-joined at triangles T_1 and T_2, respectively, to yield $G = G_1 \bigcirc G_2$.

Then G is well-covered if and only if

(1) G_1 and G_2 are both well-covered, and

(2) T_i bounds a YES-face in G_i, for $i = 1$ and 2.

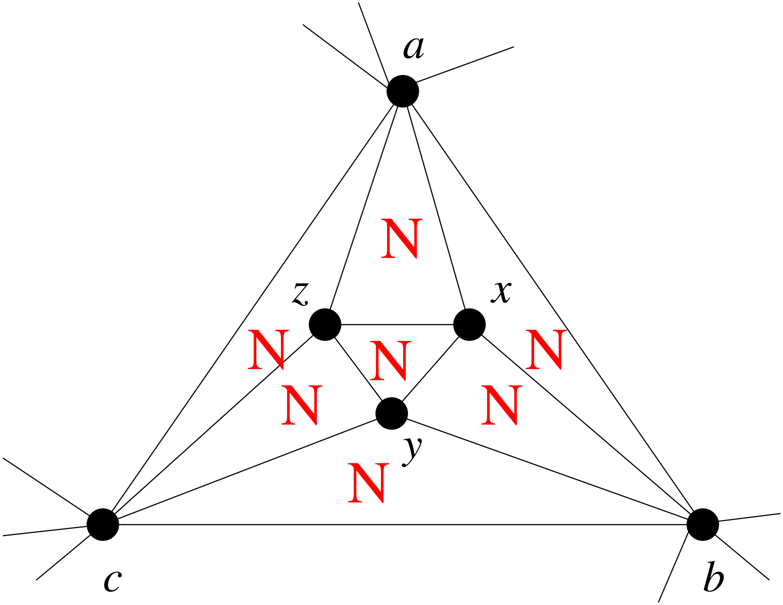
Also, if G is well-covered, then $\alpha(G) = \alpha(G_1) + \alpha(G_2)$.

SOME EXAMPLES:

(1) Both faces of K_3 are YES-faces and all faces of K_4 are YES-faces.

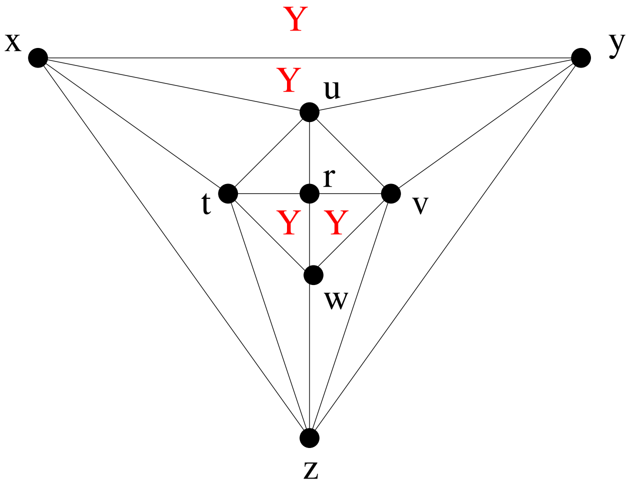
(2) If a triangle K_3 with vertices x, y and z is O-joined to a w.-c. graph G via its YES-face $abca$, to obtain a graph H, then the six faces generated in taking the O-join, together with the original K_3 form a set of seven NO-faces.

(See the next figure:)


a
z
x
y
c
b
N
N
N
N
N
N
N

(3) In R_6, R_7 and R_{12}, each triangular face is a NO-face.

(4) In R_8, the four faces labeled "Y" in the following figure are the only YES-faces.



R_8

(5) Informally, a YES-face is one at which one can O-join another w-c. triangulation and, in the process, obtain a new w-c. triangulation!

Next we consider:

Well-covered triangulations having

NO

O-joins

Def.: A vertex in a graph G is white if $\deg_G(v) = 3$ or v is adjacent to a vertex with degree 3.

(NOTE: In a well-covered triangulation, no two different K_4s can share a vertex!)

Def.: Let us call a non-white vertex blue.

At this point, we show that:

(1) If a w-c. triangulation G contains a white vertex, but *no* O-joins, then it belongs to $\mathcal{K}$; that is, *all* the vertices of G are white.

(2) If a w-c. triangulation G contains *no* white vertex and *no* O-joins, then

$$G \in \{K_3, R_7, R_8, R_{12}\}.$$

If G is a w-c. triangulation containing at least one white vertex, at least one blue vertex and has no O-joins, then we call G *bad*.

The bulk of the paper is then devoted to showing:

There is NO BAD triangulation.

This is done by considering a bad graph of minimum size.

To summarize:

Def.: The extended K_4-family, denoted $\mathcal{K}^+$, is:

(a) the collection of all graphs that can be obtained from a plane triangulation G, a member of the K_4-family $\mathcal{K}$ having at least five vertices, by choosing two disjoint sets R and S (possibly empty) of YES-faces in G and O-joining a triangle to each face in R and O-joining a copy of R_8 to each face of S via an appropriate YES-face of R_8, together with

(b) $K_4, K_4 \bigcirc K_3$ and $K_4 \bigcirc R_8$.

We can now state our characterization as follows:

Characterization Theorem: Let G be a planar triangulation. Then G is well-covered if and only if G belongs to the extended $\mathcal{K}_4$-family or else G is one of the following graphs:

$$K_3, R_6, R_7, R_8, R_{12}, R_8 \bigcirc K_3 \text{ or } R_8 \bigcirc R_8.$$

A well-covered planar triangulation is either one of ten special graphs

or it must have come from two smaller well-covered triangulations via an O-join.

One then looks for new O-joins in the two smaller component graphs and continue until the component graphs are O-join-free.

Since there can be at most a polynomial number of O-joins in a planar triangulation, we have a polynomial algorithm for recognizing planar well-covered triangulations.