Counting matrices over finite fields with zeroes on Rothe diagrams

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joint work with Aaron Klein (Brookline high school $\rightarrow$ MIT)
Joel B. Lewis (Minnesota)

$q = 1$: rook placements

$\mathfrak{S}_n = \{$permutations w of $\{1, 2, \ldots, n\}\}$

Let $S \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$

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Let $S \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$

$$p(n, S) = \#\{\text{permutations } w = w_1 \cdots w_n \mid w_i \neq j \text{ if } (i, j) \in S\}$$

$$= \#\{\text{placements of } n \text{ non-attacking rooks on } \overline{S}\}$$

$$= \#\{n \times n \text{ permutation matrices } P \mid P_{i,j} = 0 \text{ if } (i, j) \in S\}$$

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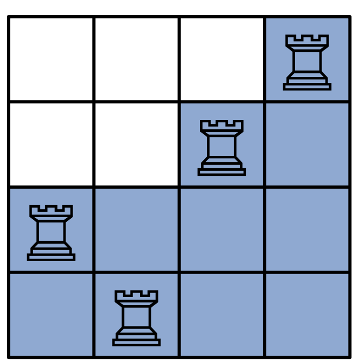
$w = 4312$



0	0	0	1
0	0	1	0
1	0	0	0
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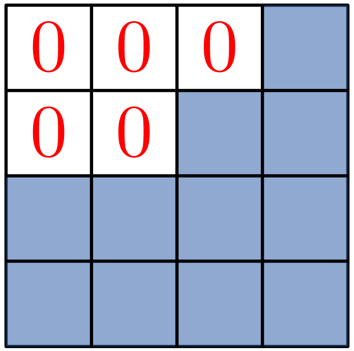
Examples

Ferrers board:

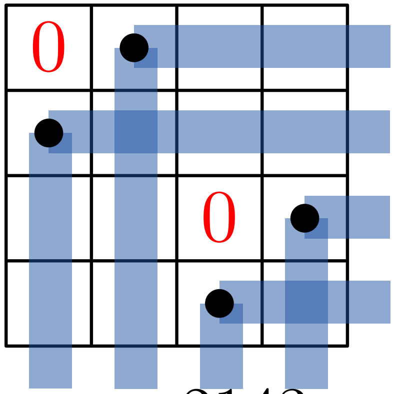
$S = \overline{F_\lambda}$



$\lambda = (1, 2, 4, 4)$

Diagram of w:

$S = D_w$



$w = 2143$

q: matrices over $\mathbb{F}_q$ with restricted support

$\mathbf{GL}(n,q) = \{n \times n$ invertible matrices over $\mathbb{F}_q\}$

$\mathbb{F}_q$ finite field $q = p^s$ elements

Let $S \subseteq \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$

$\mathrm{m}_q(n,S) = \#\{$matrices $A \in \mathbf{GL}(n,q) \mid A_{ij} = 0$ if $(i,j) \in S\}$

$q = 3$

0	0	2
1	0	1
0	2	0

0	0	1
2	1	1
1	2	0

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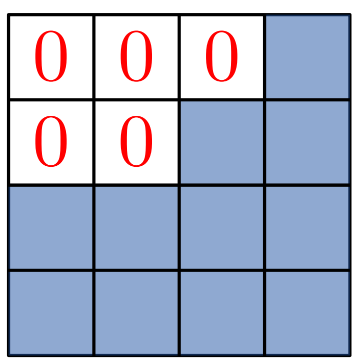

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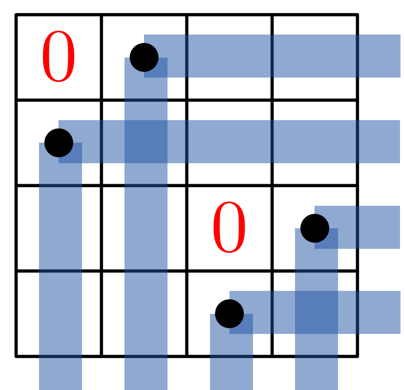

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Examples $p(n, S)$ and $\mathrm{m}_q(n, S)$

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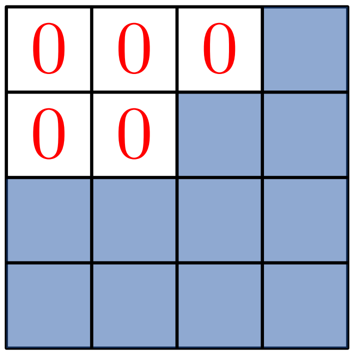

$\leftarrow$ 1 choices

$\leftarrow$ $2 - 1$ "

$\leftarrow$ $4 - 2$ "

$\leftarrow$ $4 - 3$ "

$$p(n, \overline{F_\lambda}) = \prod_{i=1}^{n} (\lambda_i - i + 1)$$

$\lambda = (1, 2, 4, 4)$ total 2 choices

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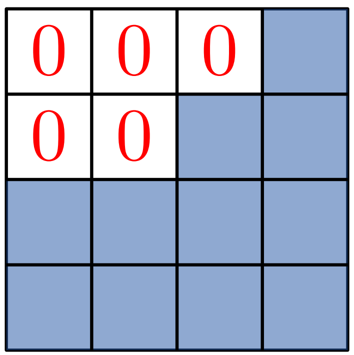

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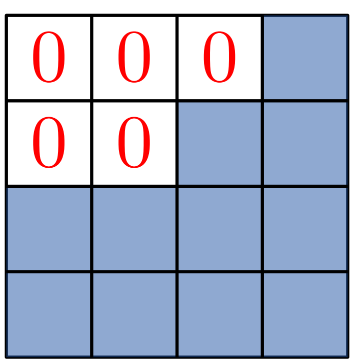

$\leftarrow$ $q^1 - 1$ choices

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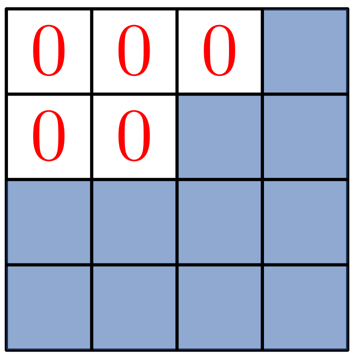

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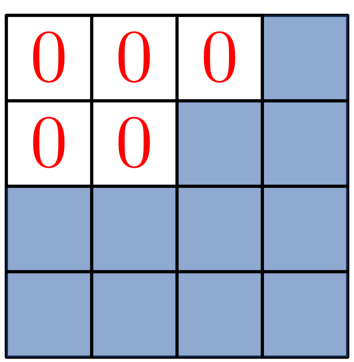

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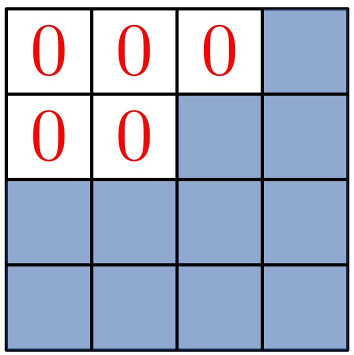

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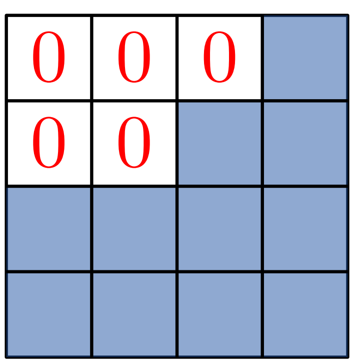

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$$\mathrm{m}_q(n, \overline{F_\lambda}) = \prod_{i=1}^{n} q^{\lambda_i} - q^{i-1}$$

$$= (q-1)^n q^{\binom{n}{2}} \prod_{i=1}^{n} [\lambda_i - i + 1]_q$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$

Remarks and outline

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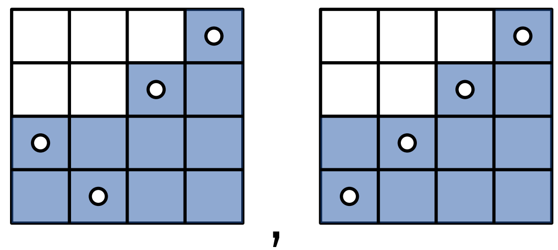
  ii) $p(n, \overline{F}_\lambda) = \#\{$  $\}$

an interval in the Bruhat order

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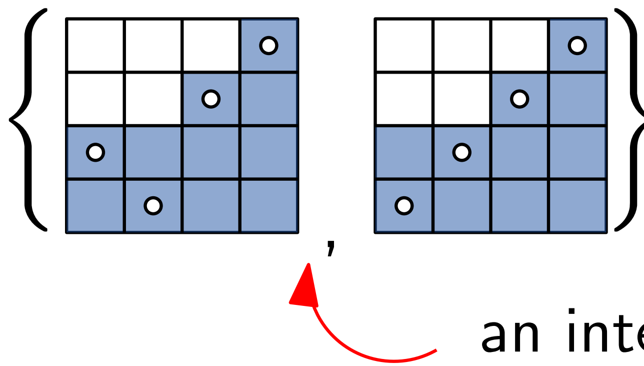
  ii) $p(n, \overline{F}_\lambda) = \#$ 

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Outline

I Is $\mathrm{m}_q(n, S)/(q-1)^n$ a polynomial in q? Is it in $\mathbb{N}[q]$?

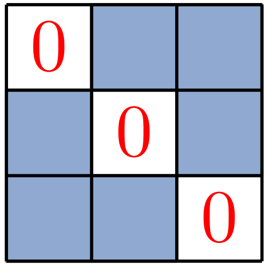
II When are rook placements on $\overline{S}$ related to Bruhat intervals?

Is $\mathrm{m}_q(n, S)/(q-1)^n$ a polynomial in $\mathbb{N}[q]$?

Example

If $S = \{(1,1),\ (2,2),\ (3,3)\}$,

$\mathrm{m}_q(3, S) = (q-1)^3(q^3 + 2q^2 - q)$



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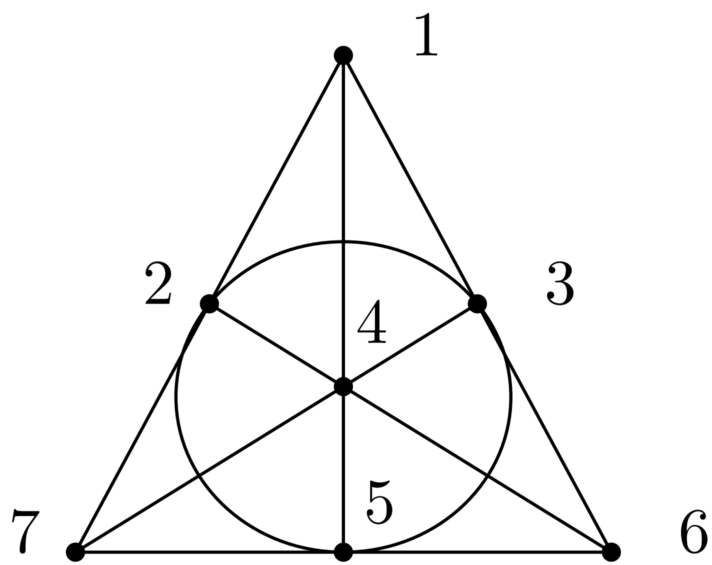
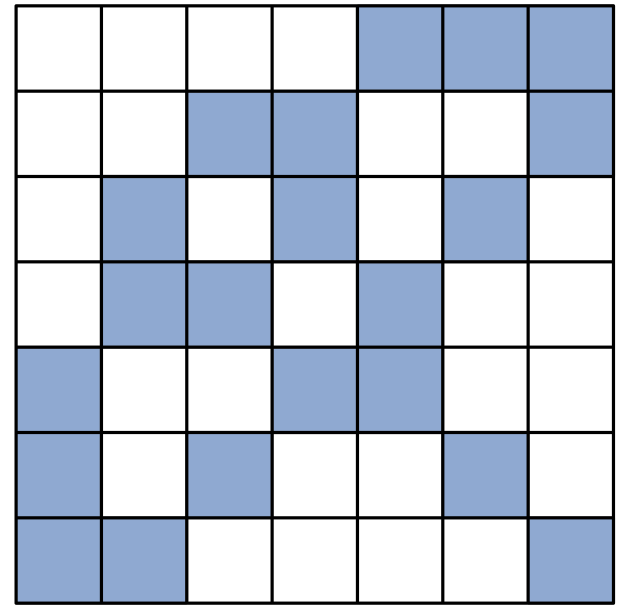
Example (Stembridge 1998)



Fano plane $PG(2,2)$



$S_{PG(2,2)}$

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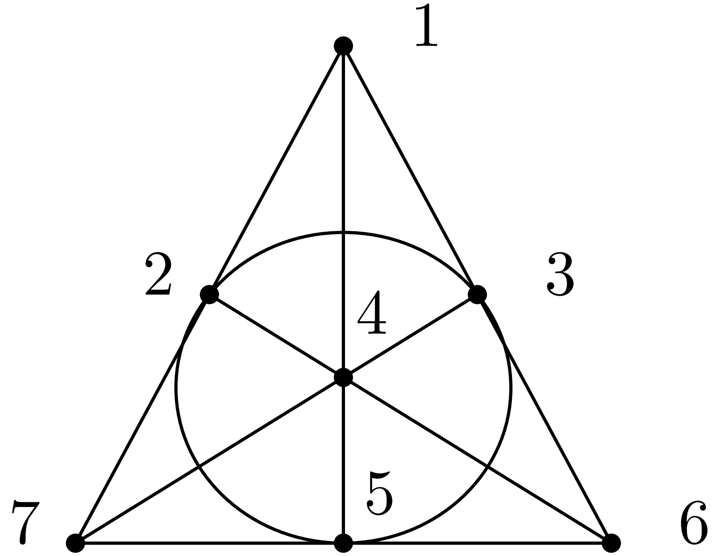

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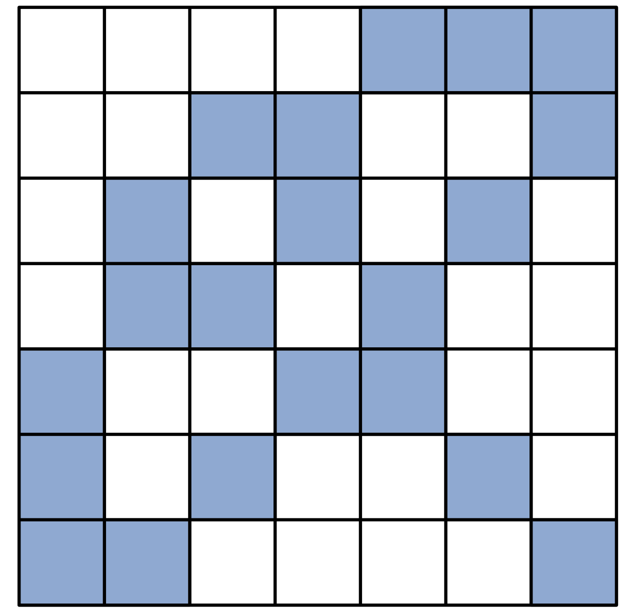

$S_{PG(2,2)}$

$$\mathrm{m}_q(7, S_{PG(2,2)}) = \begin{cases} (q-1)^7(q^{14} + \cdots - 97q^9 + \cdots + q^3) & \text{if } q \text{ even,} \\ (q-1)^7(q^{14} + \cdots - 98q^9 + \cdots - 6q^5) & \text{if } q \text{ odd.} \end{cases}$$

however, $\mathrm{m}_q(n,S)$ is a q-analogue of $p(n,S)$

Theorem (Lewis, Liu, M, Panova, Sam, Zhang 2011)

For **all** $S \subset \{1,\ldots,n\} \times \{1,\ldots,n\}$,

$$\mathrm{m}_q(n,S) \equiv p(n,S) \cdot (q-1)^n \pmod{(q-1)^{n+1}}.$$

- If $S = \{(1,1),\,(2,2),\,(3,3)\}$,

$$\mathrm{m}_q(3,S)/(q-1)^3 = q^3 + 2q^2 - q$$
$$\mathrm{m}_q(3,S)/(q-1)^3\Big|_{q=1} = 2$$

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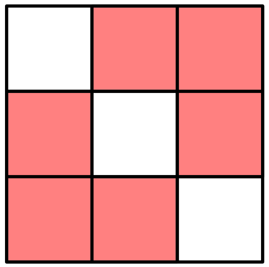

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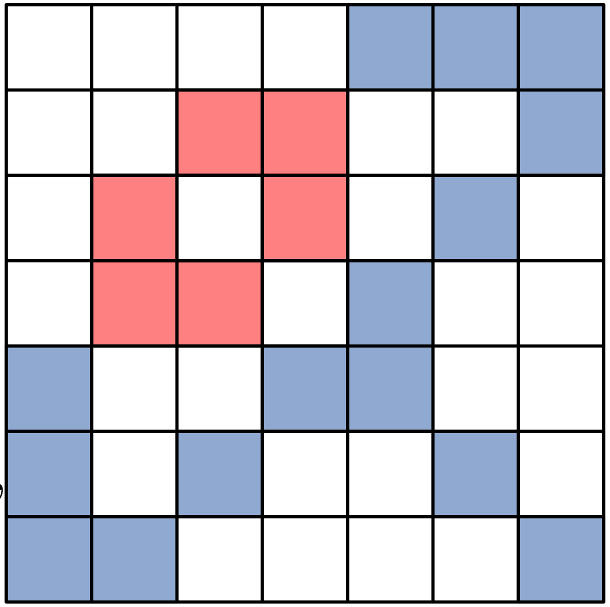

Outline of talk

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For which S is it in $\mathbb{N}[q]$?

– If S is a Ferrer's board: F_λ ?

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II When are rook placements on $\overline{S}$ related to Bruhat intervals?

$\mathrm{m}_q(n, \overline{F_\lambda})/(q-1)^n$ is in $\mathbb{N}[q]$

Theorem (Haglund 1998)

$$\mathrm{m}_q(n, \overline{F_\lambda}) = (q-1)^n \sum_{w \in p(n, \overline{F_\lambda})} q^{\mathrm{boxNE}(w)}.$$

boxes ■ in F_λ strictly North and East of rooks

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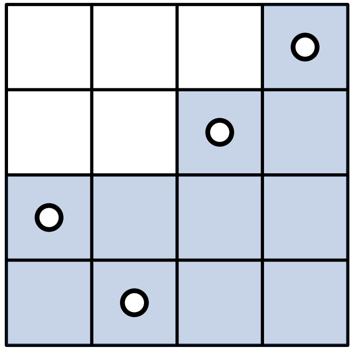

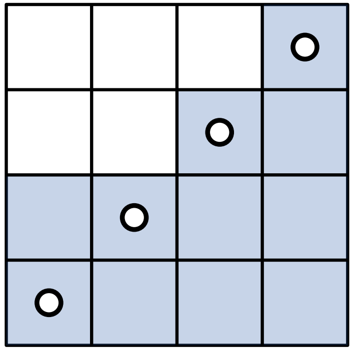

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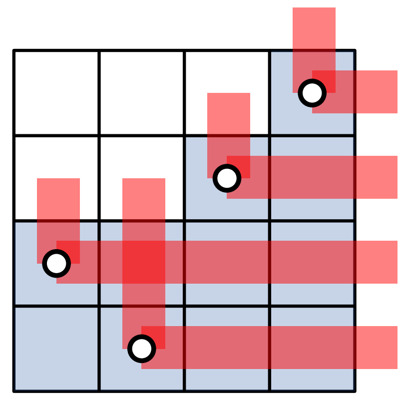

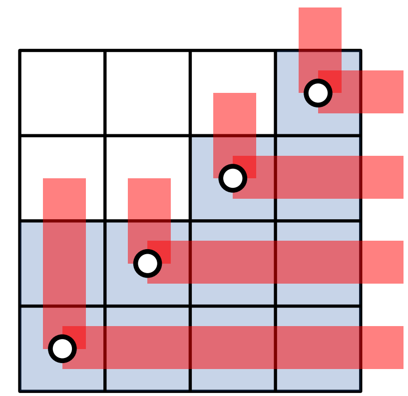

$$\mathrm{m}_q(4, \overline{F_{1244}}) = (q-1)^4 (q^6 + q^7)$$

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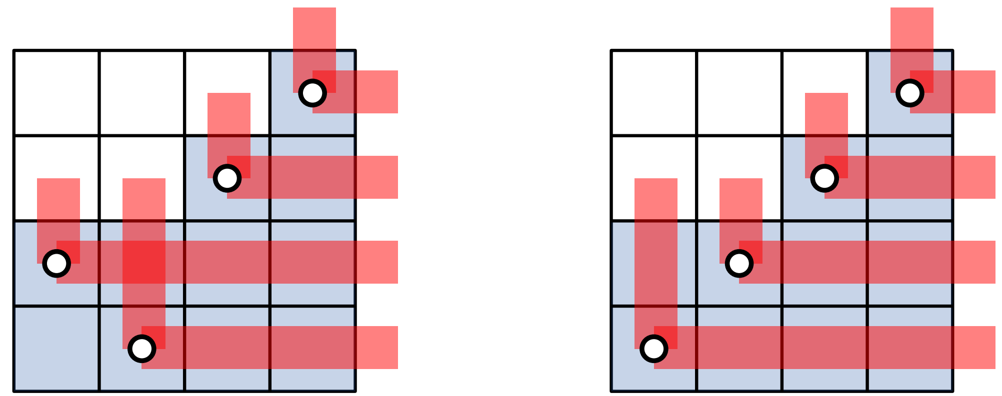

$$\mathrm{m}_q(4, \overline{F_{1244}}) = (q-1)^4 (\quad q^6 \quad + \quad q^7 \quad)$$

Proposition (Klein-Lewis-M 2012)

Haglund's result extends to **skew** Ferrers shapes $S = \overline{F_{\lambda/\mu}}$

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boxes in F_λ strictly North and East of rooks

Example

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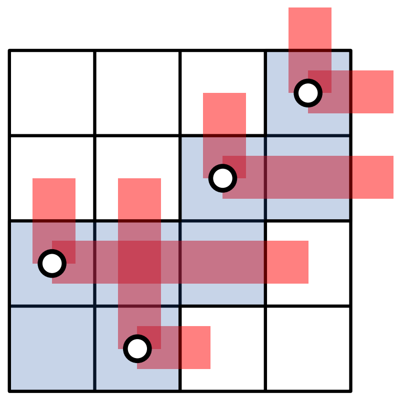
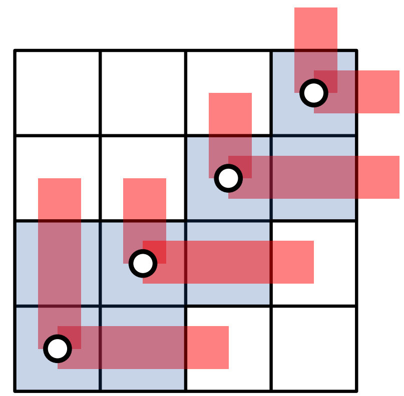
$\mu = (1, 2)$





$$\mathrm{m}_q(4, \overline{F_{1244/12}}) = (q-1)^4 (\quad q^3 \quad + \quad q^4 \quad)$$

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$\mathrm{m}_q(n, S)$ when S is the diagram of a permutation: D_w

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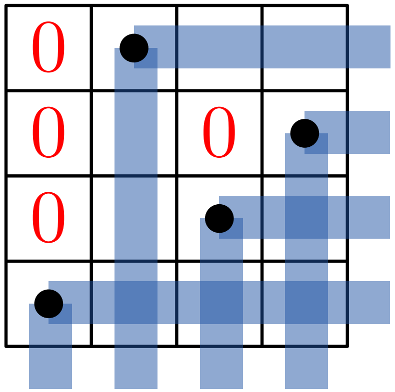
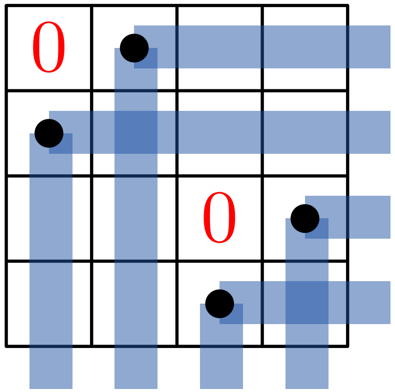


$w = 2431$

$D_{2431} = \{(1,1), (2,1), (3,1), (2,3)\}$



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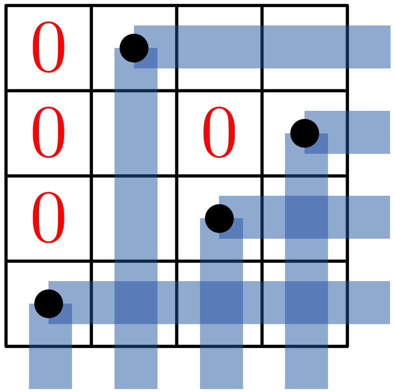
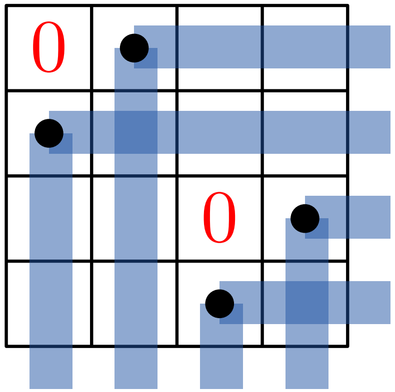
$\mathrm{m}_q(n, S)$ when S is the diagram of a permutation: D_w

Examples



$w = 2431$

$D_{2431} = \{(1,1), (2,1), (3,1), (2,3)\}$



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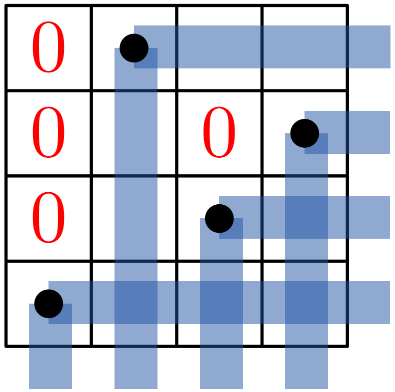

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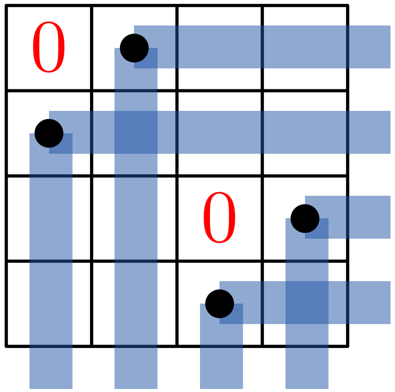

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Can we prove the conjecture for some families of permutations?

Vexillary permutations

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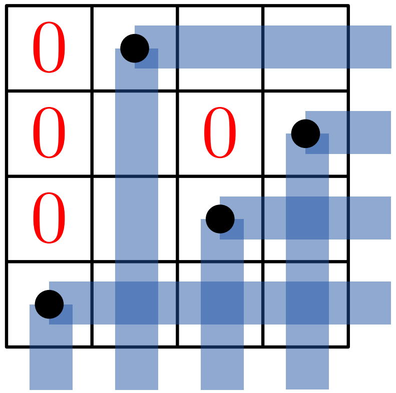

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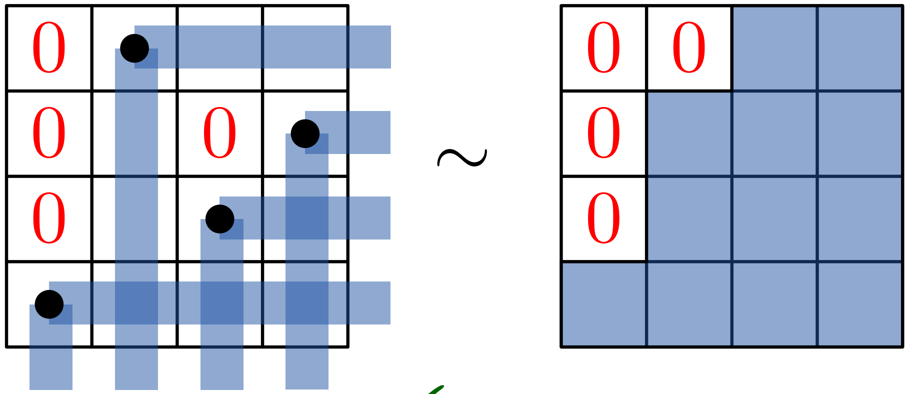

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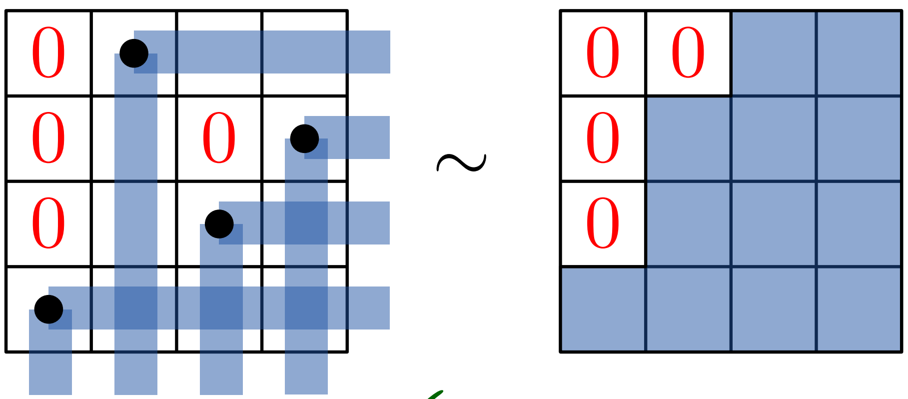

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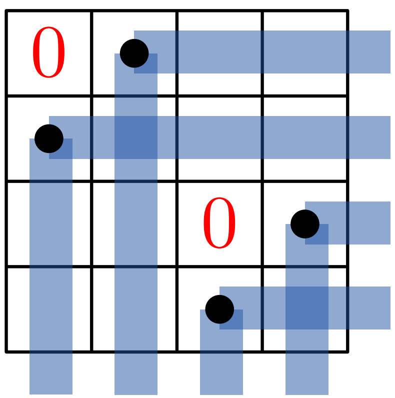

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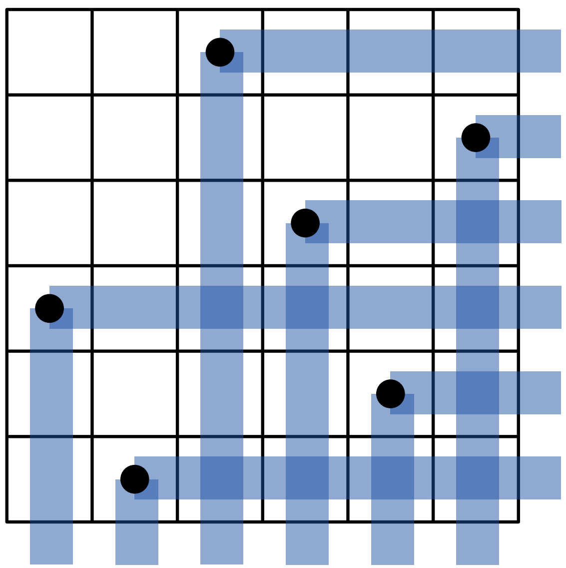

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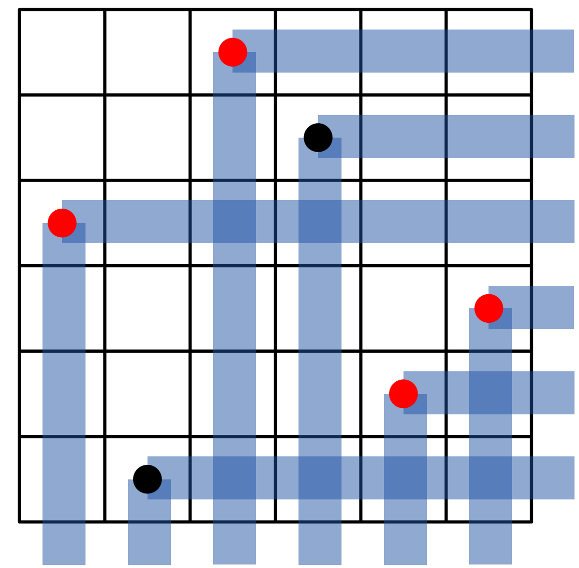

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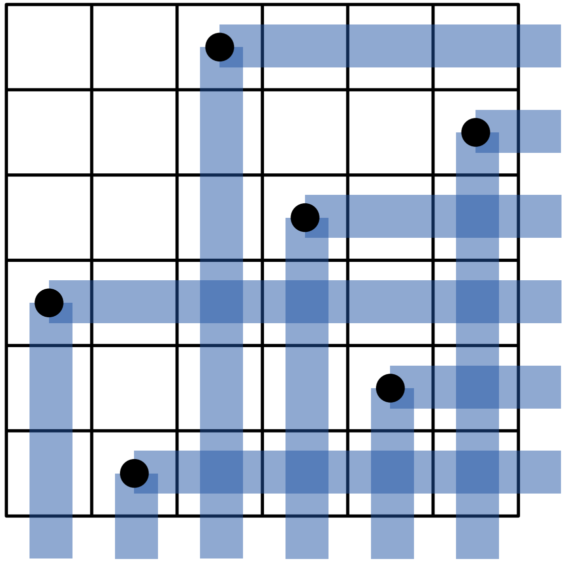

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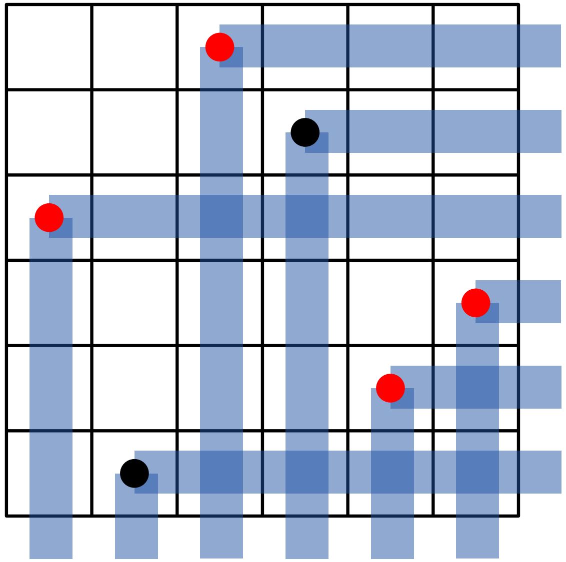

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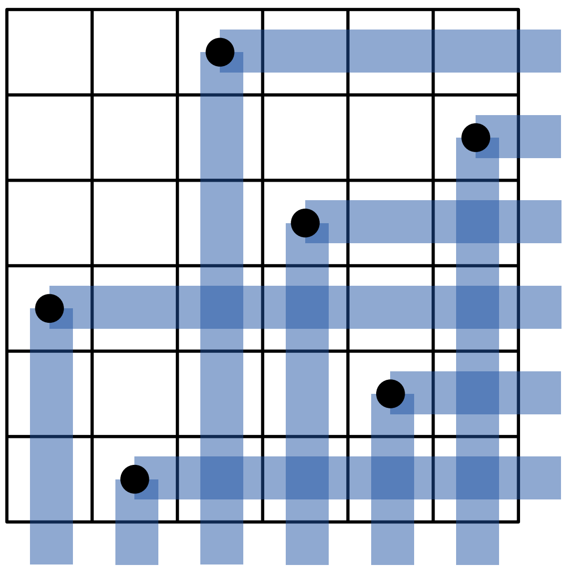

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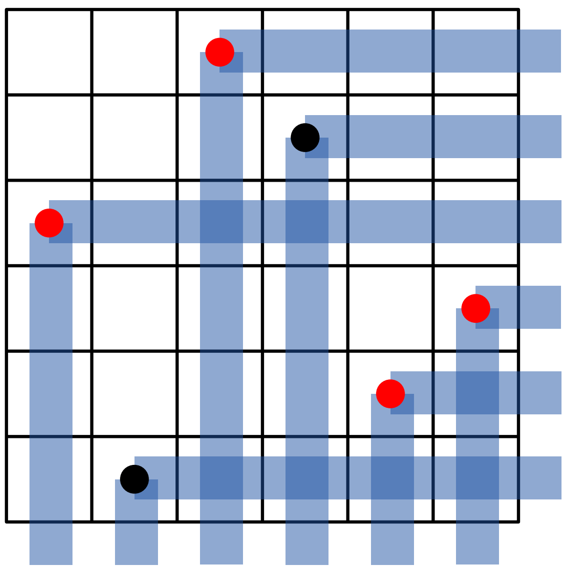

- such permutations are called **vexillary**
- there is characterization of w such that D_w is the complement of a skew Ferrers shape (Klein-Lewis-M 2012)

Outline of talk

I Is $\mathrm{m}_q(n,S)/(q-1)^n$ a polynomial in q?, Is it in $\mathbb{N}[q]$? ✗

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Rook placements on F_λ and Bruhat intervals

(strong) Bruhat order ($\preceq$) on $\mathfrak{S}_n$:

$$u \prec u \cdot (i,j) \quad \text{if} \quad \mathrm{inv}(u \cdot (i,j)) = \mathrm{inv}(u) + 1$$

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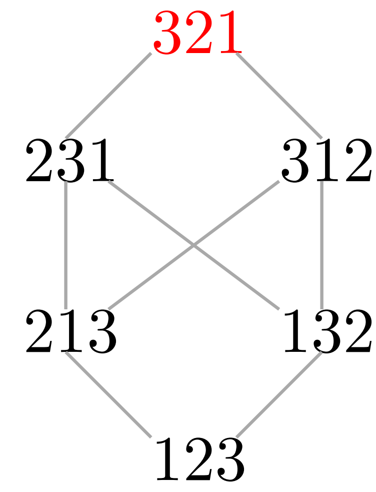
Example ($n = 3$)



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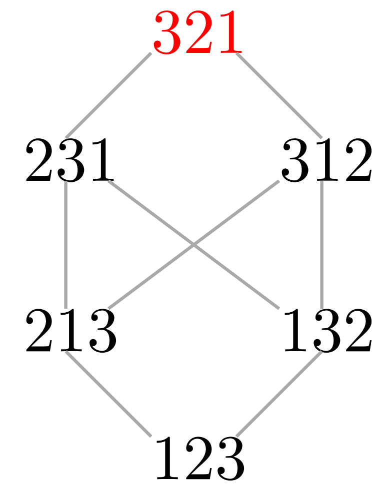

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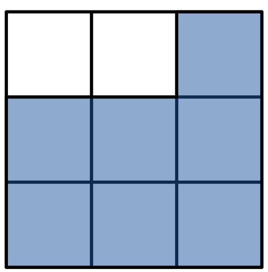
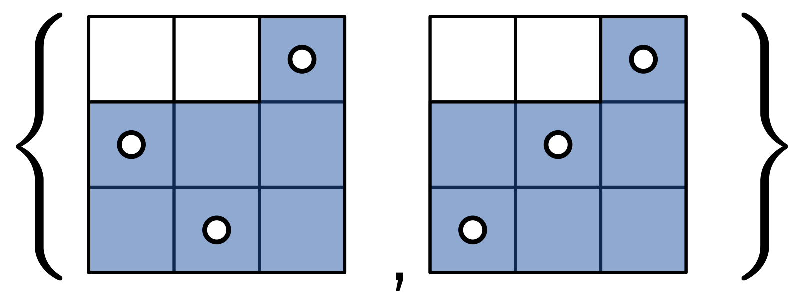


$\lambda = (1,3,3)$

3-rook placements =



312 , 321

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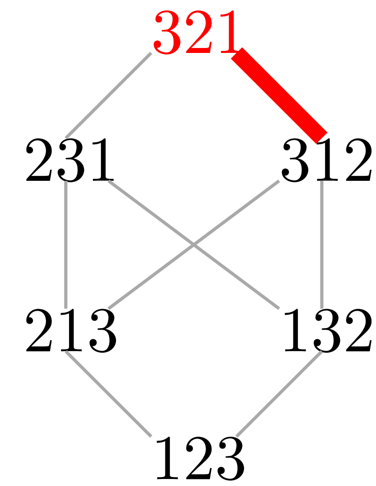

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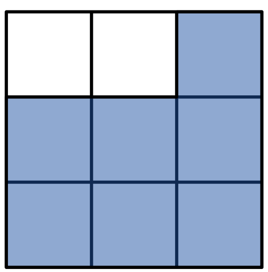

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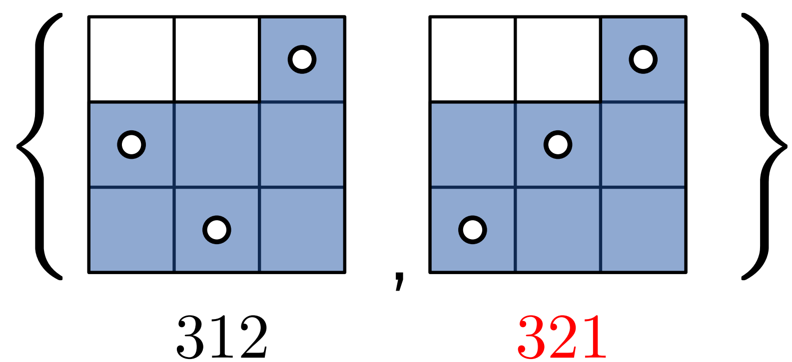

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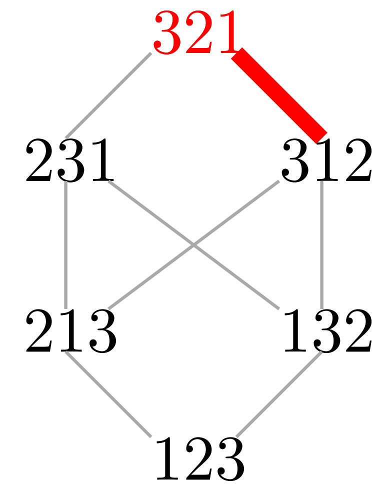

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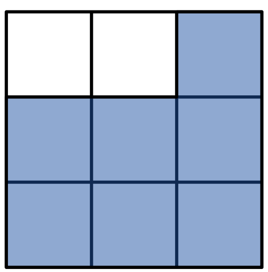

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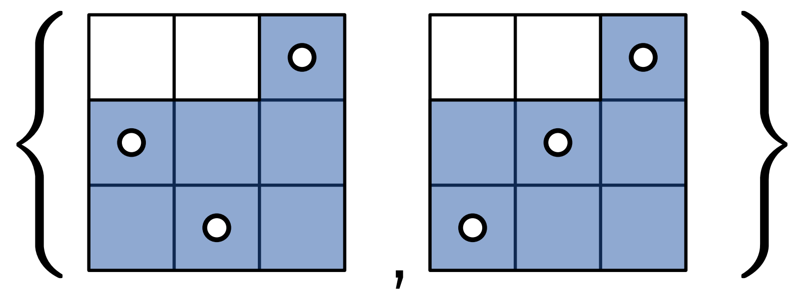

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Theorem (Ding 2001)

$\{n$-rook placements on $F_\lambda\} = [w_\lambda, n\, n-1 \ldots 21]$ for some 213-avoiding permutation w_λ.

- $p(n, \overline{F_\lambda}) = \#[w_\lambda, n\, n-1 \ldots 21]$

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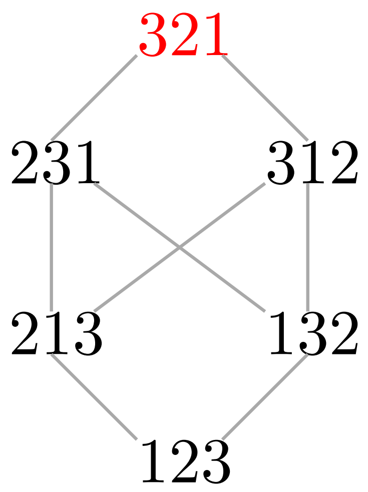

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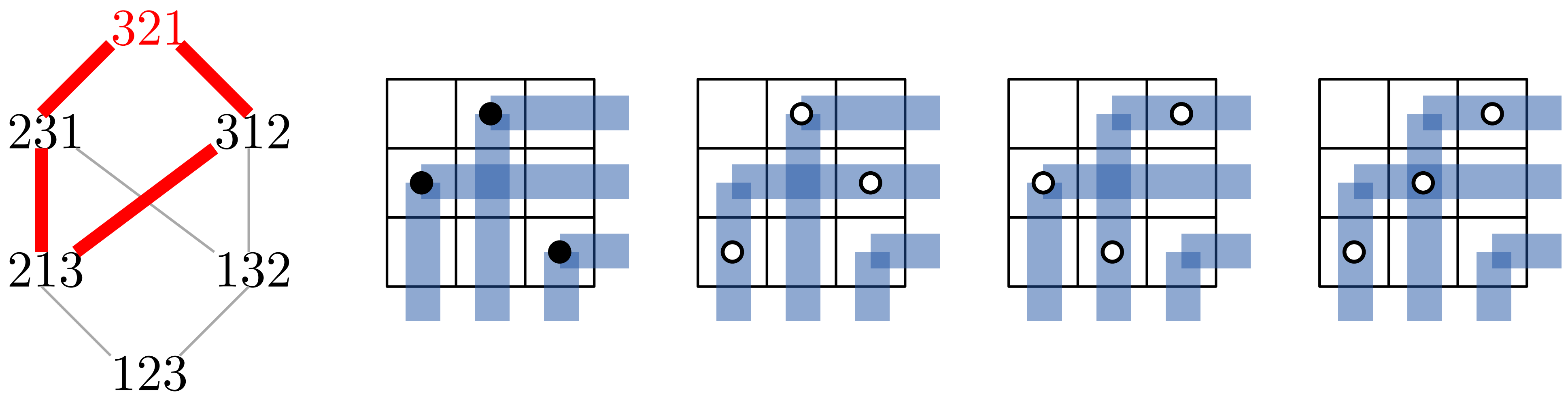

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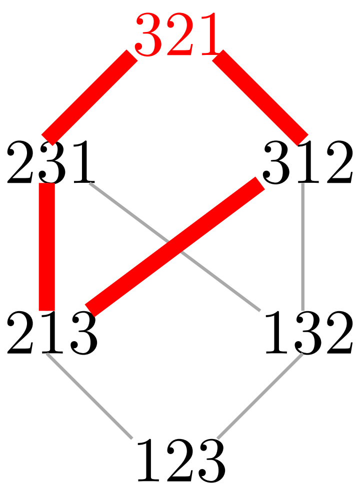
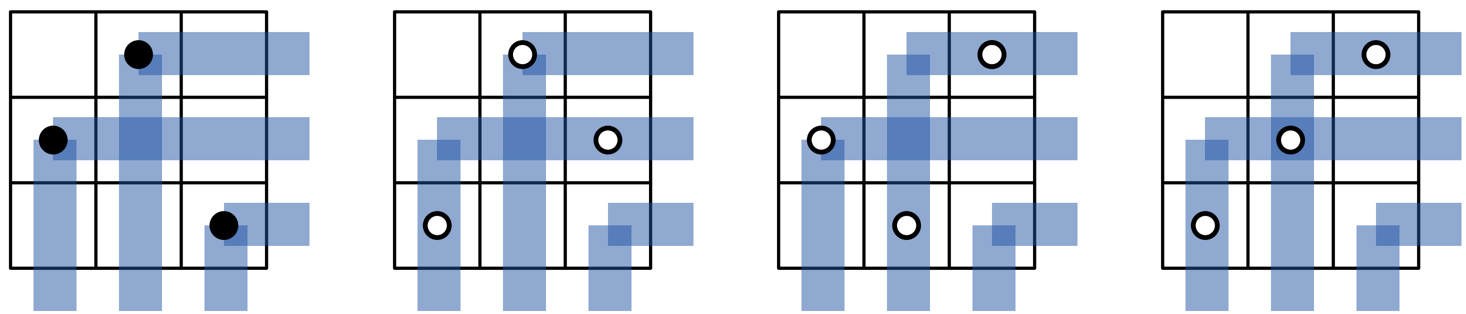
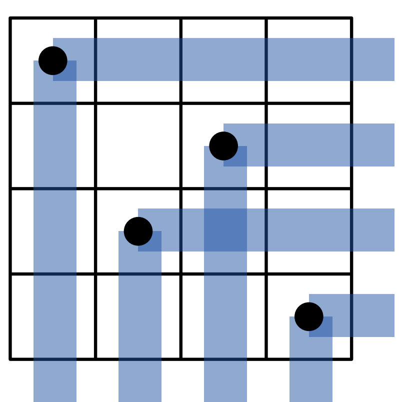




$w = 1324$



$\#[1324, 4321] = 20$

$p(4, D_{1324}) = 18$

q: $\mathrm{m}_q(n, D_w)$ and $\#[w, n\,n-1\cdots 21]$

Conjecture (Lewis-Klein-M 2012)

For all w,

$$q^{\binom{n}{2}-\mathrm{inv}(w)} \sum_{u \succeq w} q^{\mathrm{inv}(u)} - \mathrm{m}_q(n, D_w)/(q-1)^n \in \mathbb{N}[q],$$

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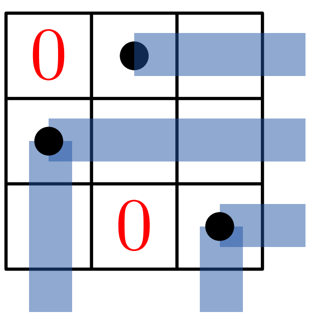
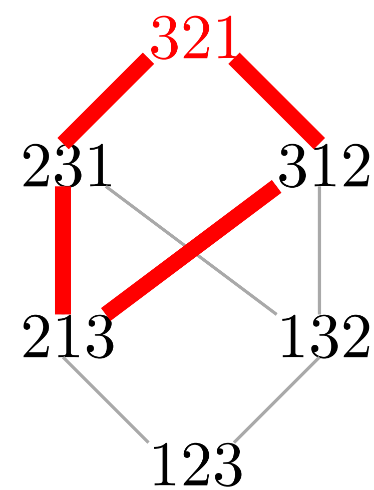


$$\frac{\mathrm{m}_q(3, D_{213})}{(q-1)^3} = q^2(q^3 + 2q^2 + q^1)$$



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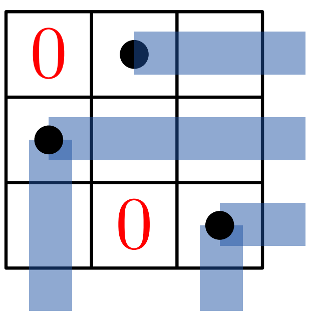

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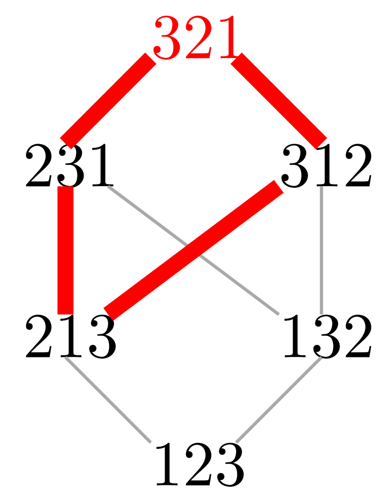

Remark

- same patterns appear in: Gasharov-Reiner 02, Postnikov 06, Sjöstrand 07.

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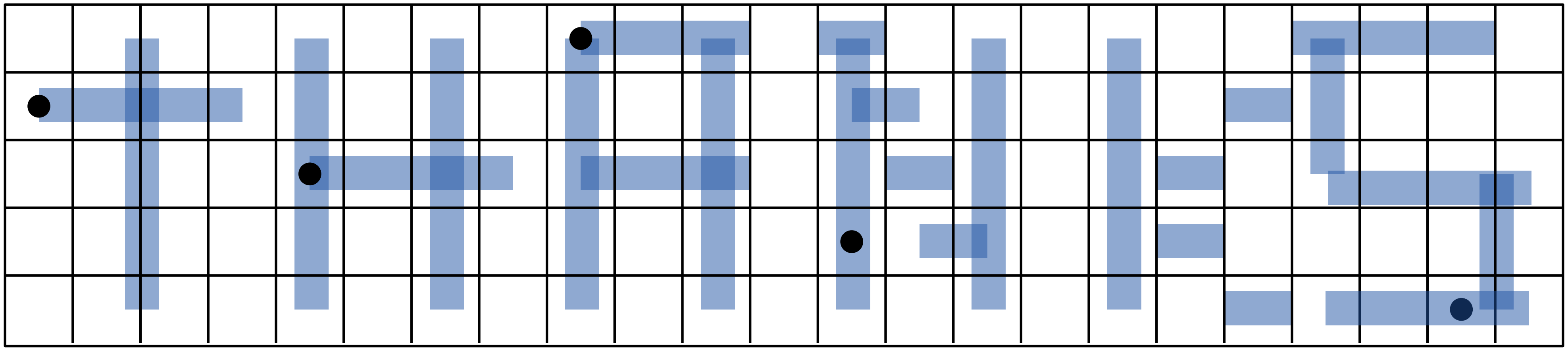

Bibliography:

- with A. Klein, J. Lewis, *Counting matrices over finite fields with support on skew Young and Rothe diagrams*, **arXiv:1203.5804**
- with J. Lewis, R. Liu, G. Panova, S. Sam, Y. Zhang, *Matrices with restriced entries and q-analogues of permutations*, **arXiv:1011.4539**

Code (sage and maple):

`https://sites.google.com/site/matrixfinitefields/`