Clusters of solutions to random linear equations

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includes work with Dimitris Achlioptas and with Jane Gao

A Random System of Linear Equations

We have $M = cn$ linear equations over n $\{0, 1\}$ variables.

All addition is $\mod 2$.

$$\begin{aligned} x_5 + x_1 + x_6 &= 0 \\ x_2 + x_6 + x_1 &= 1 \\ x_1 + x_2 + x_5 &= 1 \\ x_3 + x_5 + x_4 &= 0 \\ x_6 + x_4 + x_2 &= 1 \end{aligned}$$

Each equation contains

- a random k-tuple of variables
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Also known as k-XORSAT.

Random Constraint Satisfaction Problems

This is one of the standard models of random constraint satisfaction problems.

It is one of the simplest of the commonly studied models:

- lots of symmetry
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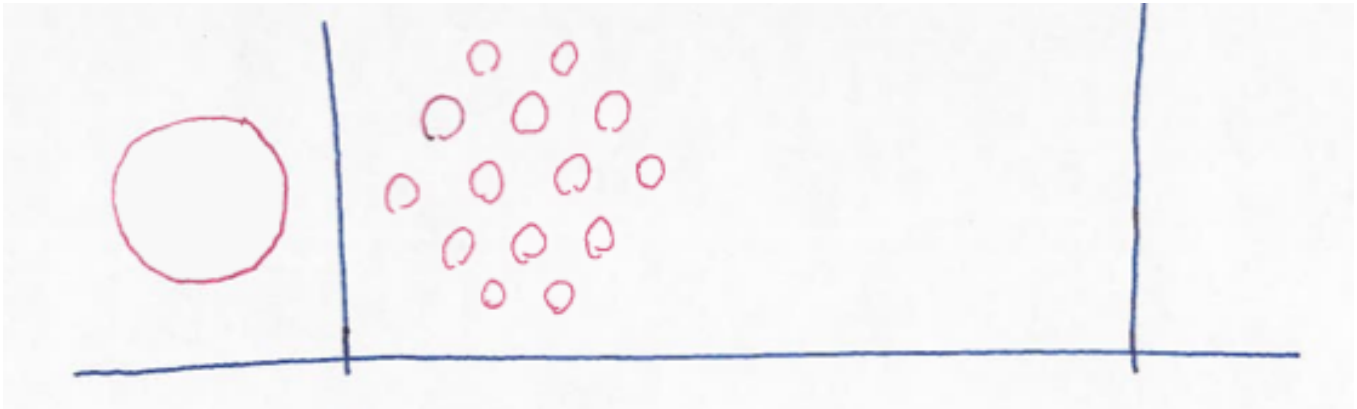
Satisfiability Threshold:

$$c = .917..., \qquad k = 3 \qquad \text{Dubois and Mandler, 2002}$$

$k > 3$ Dietzfelbinger, et al, 2010, Pittel and Sorkin, 2012.

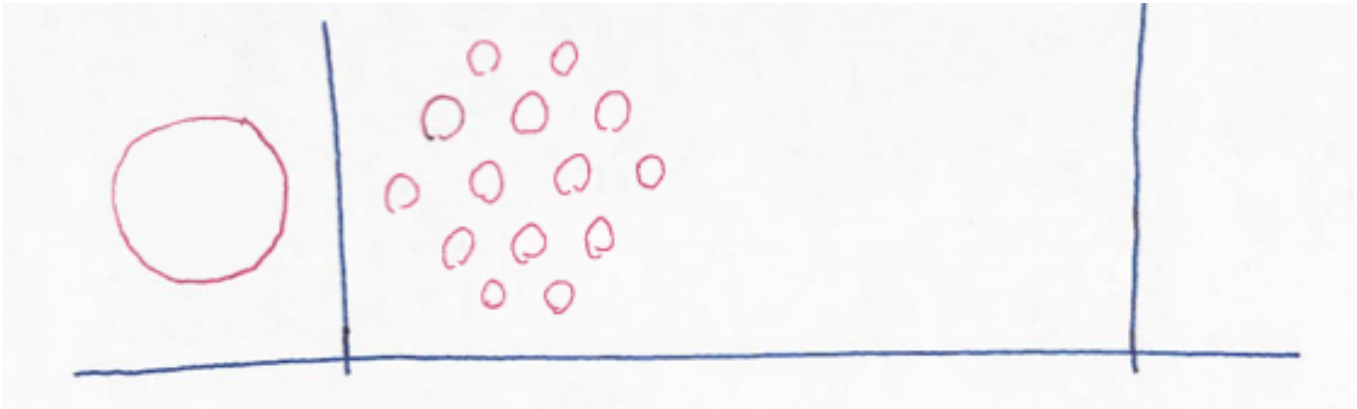
Clustering



c_c c_s

Phenomenon seems to hold for a wide variety of random CSP's.

Clustering



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- Well-connected. One can move throughout the cluster changing $o(n)$ variables at a time.
- Well-separated Moving from one cluster to another requires changing $\Theta(n)$ variables in one step.

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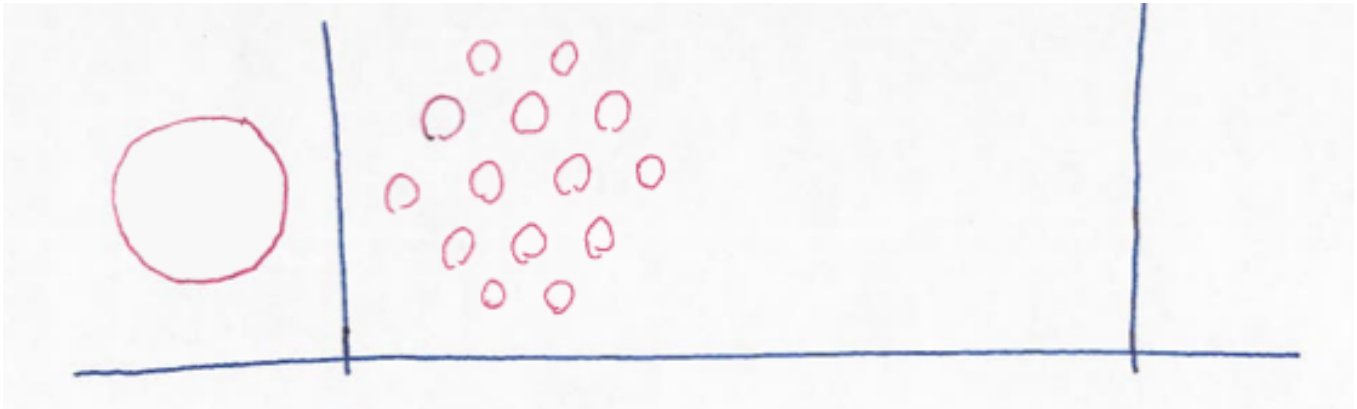

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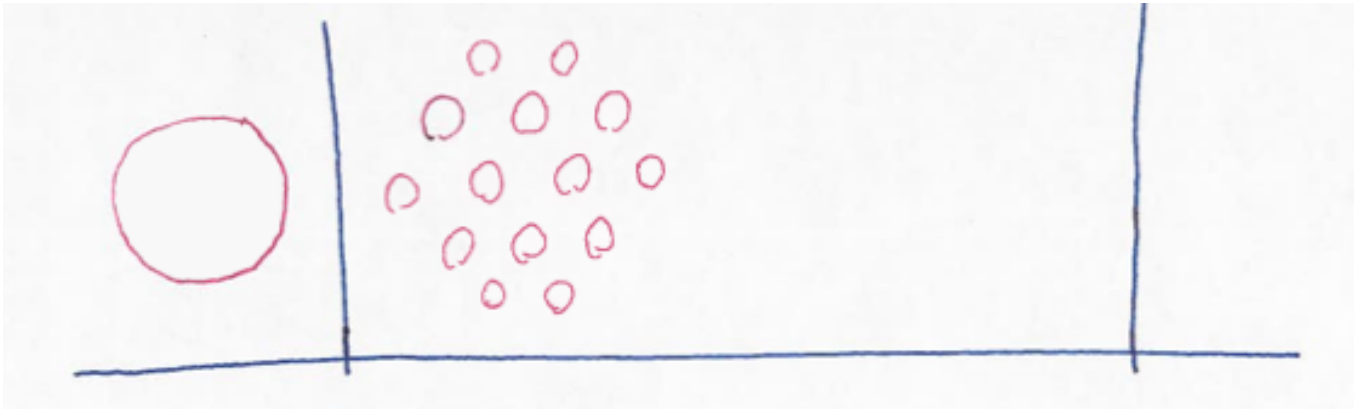
This is mostly non-rigorous, but:

- It is based on some substantial mathematical analysis
- It explains a lot
  - earlier results
  - algorithmic challenges
- "Knowing" that it is true can inspire proof approaches (eg. the previous talk)
- Understanding random CSP's will require understanding clustering

Clustering



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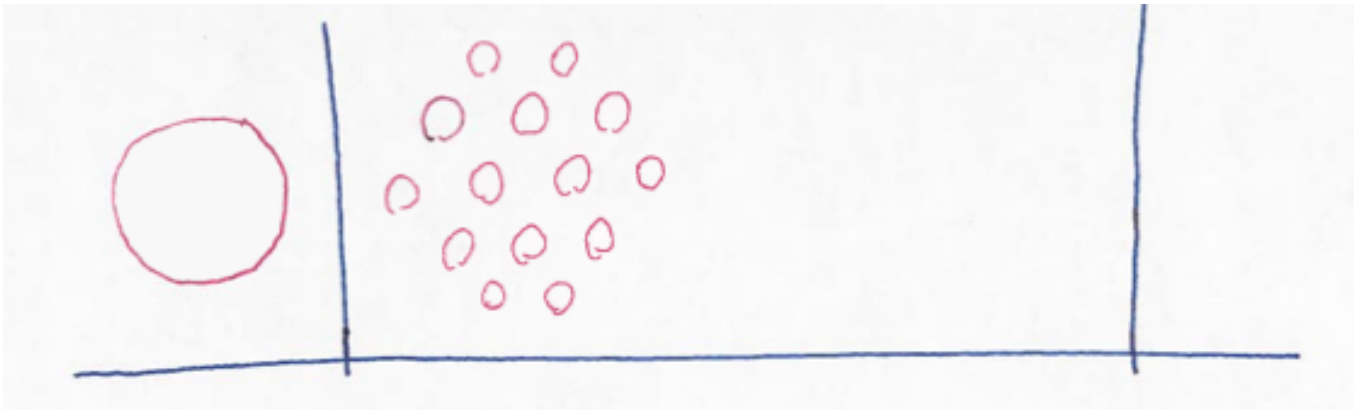

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Ibrahimi, Kanoria, Kranning and Montanari (2011)
Achlioptas and M (2011)

The 2-core

Remove every variable that appears in at most one equation, along with the equation it belongs to.

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Reason: Every solution to what remains can easily be extended to the original system, by setting the deleted variable.

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Iterate

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This is also the 2-core of the underlying hypergraph:

- vertices are the variables
- hyperedges are the k-tuples of vertices that form equations

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The satisfiability threshold is the point where the 2-core has density 1.

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Clusters:

- σ is any solution to the 2-core.
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Technical correction: We actually need to work with the 2-core minus $O(1)$ variables because of short cycle effects.

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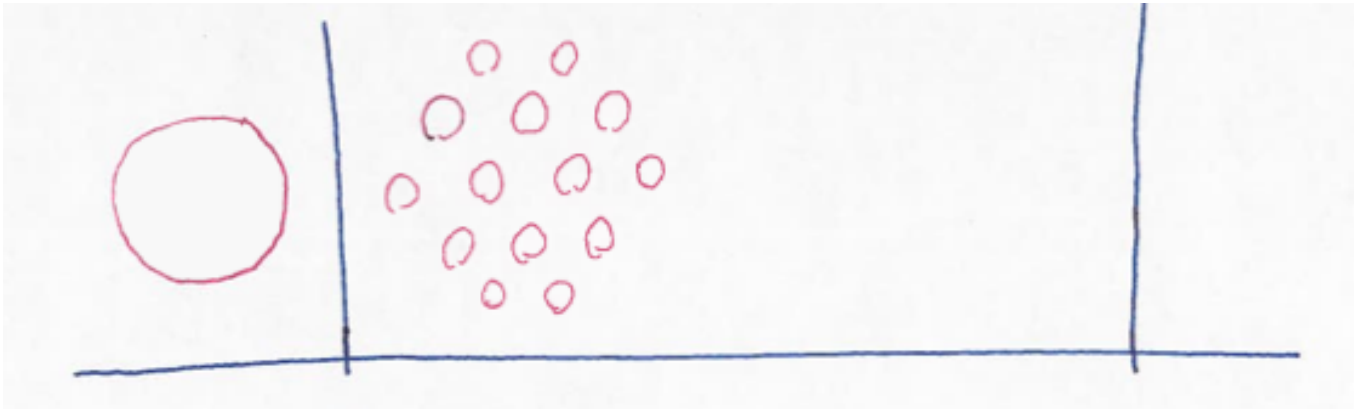

Roughly speaking, clusters are:

- Well-connected. One can move throughout the cluster changing $o(n)$ vertices at a time.
- Well-separated Moving from one cluster to another requires changing $\Theta(n)$ vertices in one step.

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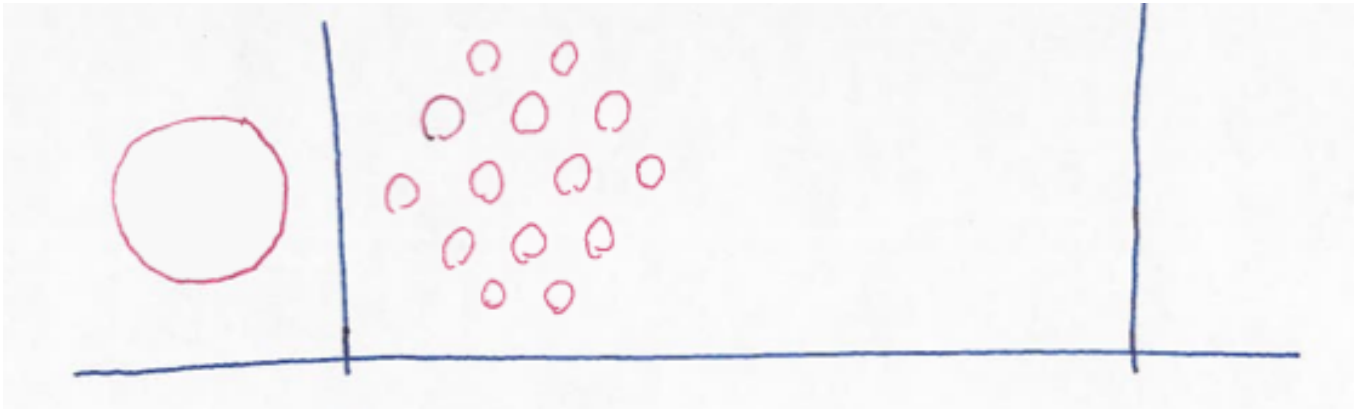

Theorem (IKKM, AM)

- *Any pair of 2-core solutions must differ on at least αn variables.*
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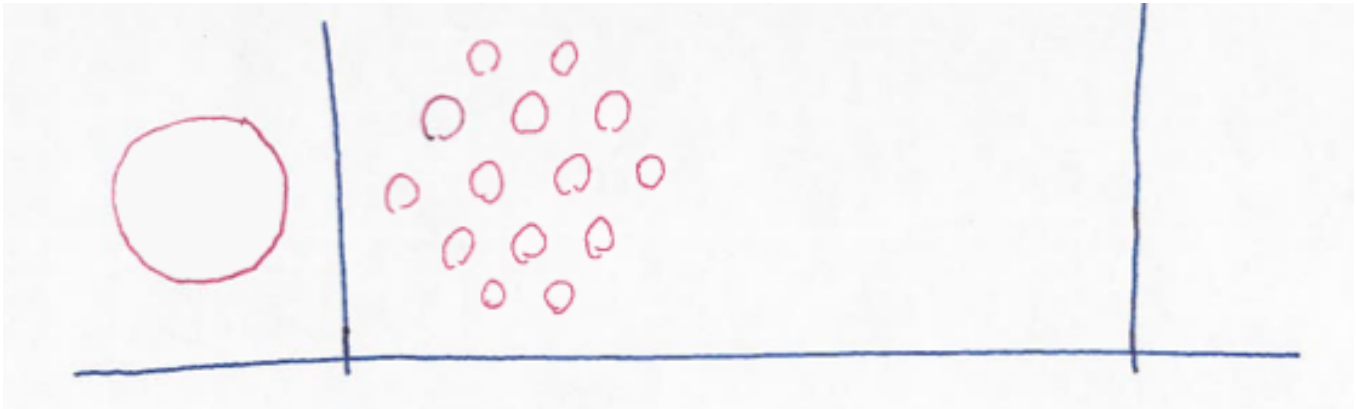

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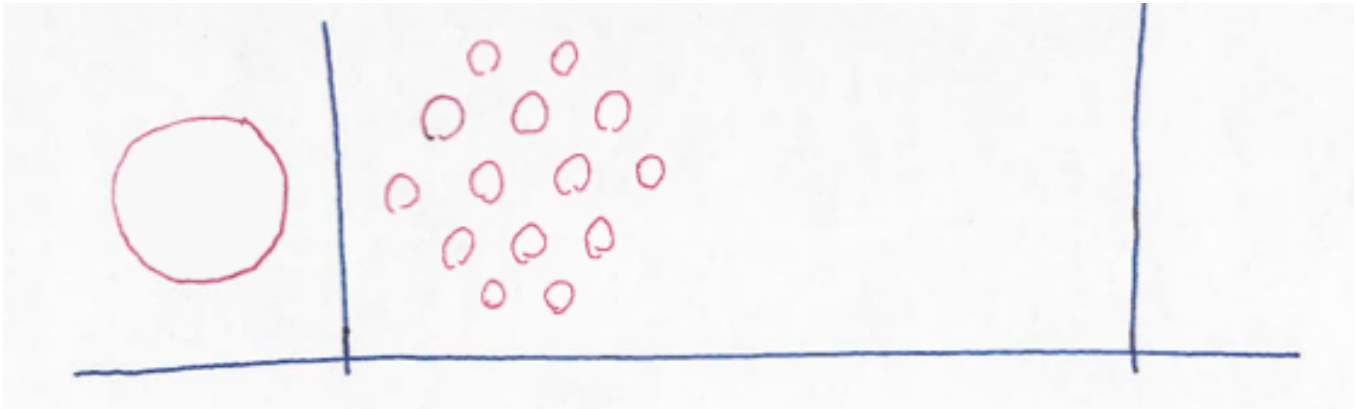

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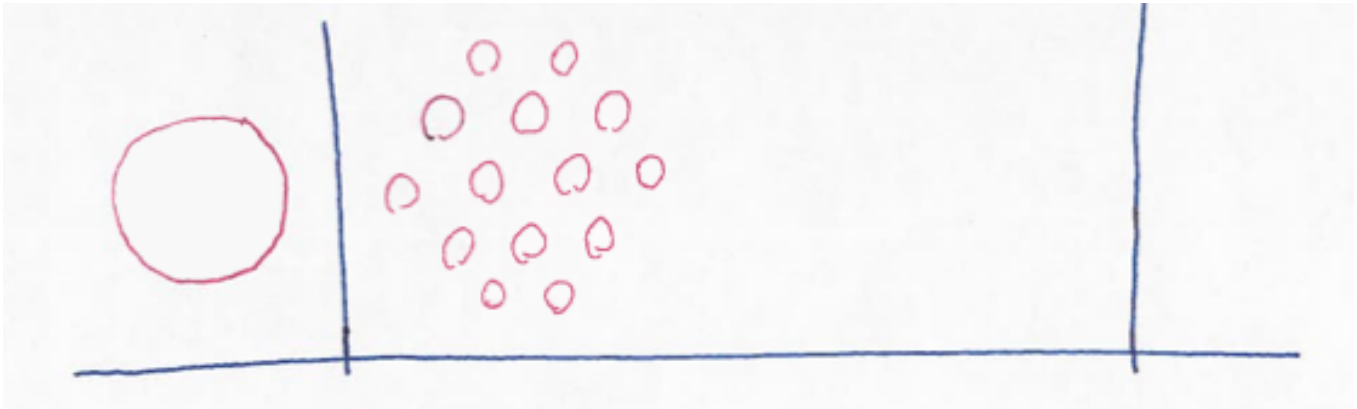

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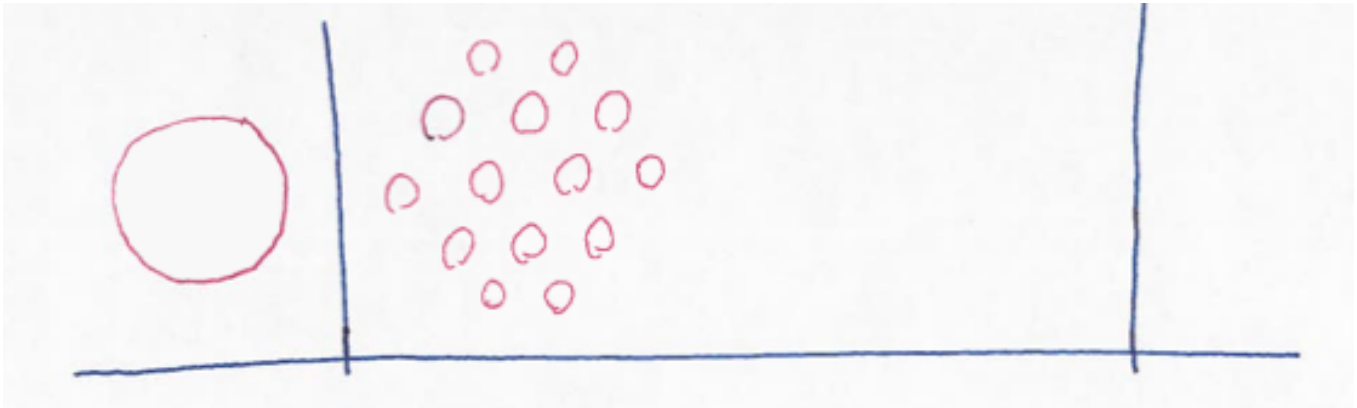

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- The same variables are frozen in every cluster.
- No condensation.
- We only need to analyze the random hypergraph, not actual solutions of the CSP.

Proven For Other CSPs

Well-Separated Clusters: Asymptotic in k threshold for

- k-SAT, k-colouring, hypergraph 2-colouring Achlioptas and Coja-Oghlin 2008
- independent set Coja-Oghlin and Efthymiou 2010
- several others Montanari, Restrepo, Tetali 2009

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Freezing:

- occurs in k-SAT Achlioptas and Ricci-Tersinghi 2006
- Asymptotic in k threshold for k-SAT, k-colouring, hypergraph 2-colouring Achlioptas and Coja-Oghlin 2008
- exact threshold for k-colouring M 2011
- exact threshold for hypergraph 2-colouring and others M and Restrepo 2013

Gaussian Elimination

Stripping Digraph: When we remove x, we direct an edge from x to each of the other $k-1$ variables in its equation.

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Lemma

Each non 2-core x_j is reachable from $O(\log n)$ other variables.

So we can move between solutions in C_σ by changing base variables one at a time. Each change affects $O(\log n)$ variables.

Depth in r-cores

r-core of a hypergraph: Repeatedly remove vertices of degree $< r$.

Note that the order in which vertices are removed does not affect the core obtained.

This is the largest subgraph with minimum degree at least r.

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Analyzed for random graphs in Pittel, Spencer, Wormald(1996) and many other papers.

Applications include colouring, hashing, coding, orientability,...

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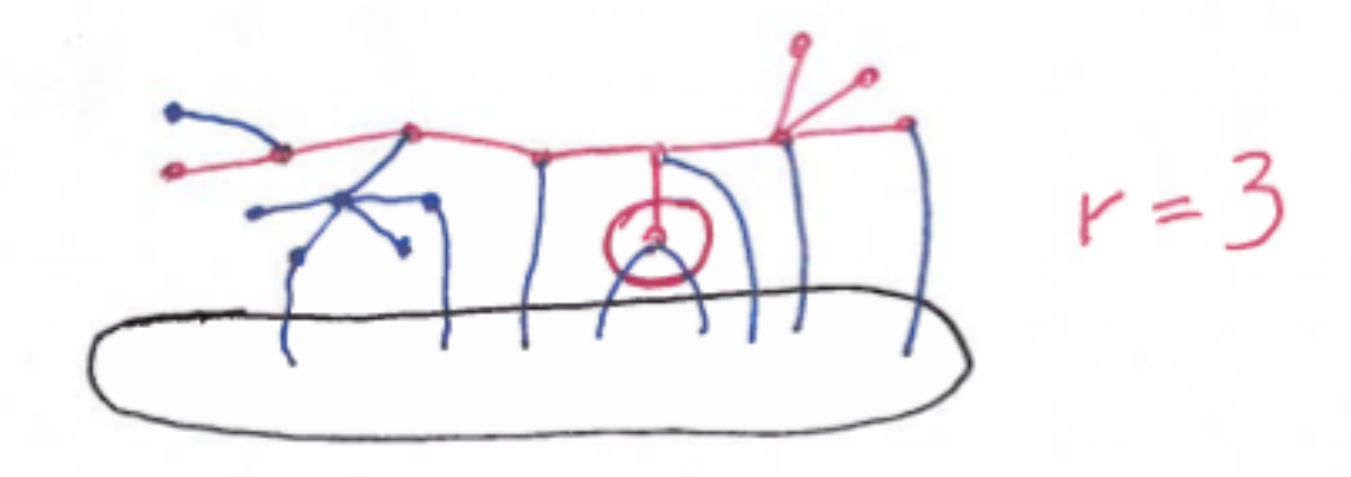
The depth of a non r-core vertex v is the shortest sequence of deletions that leads to removing v.



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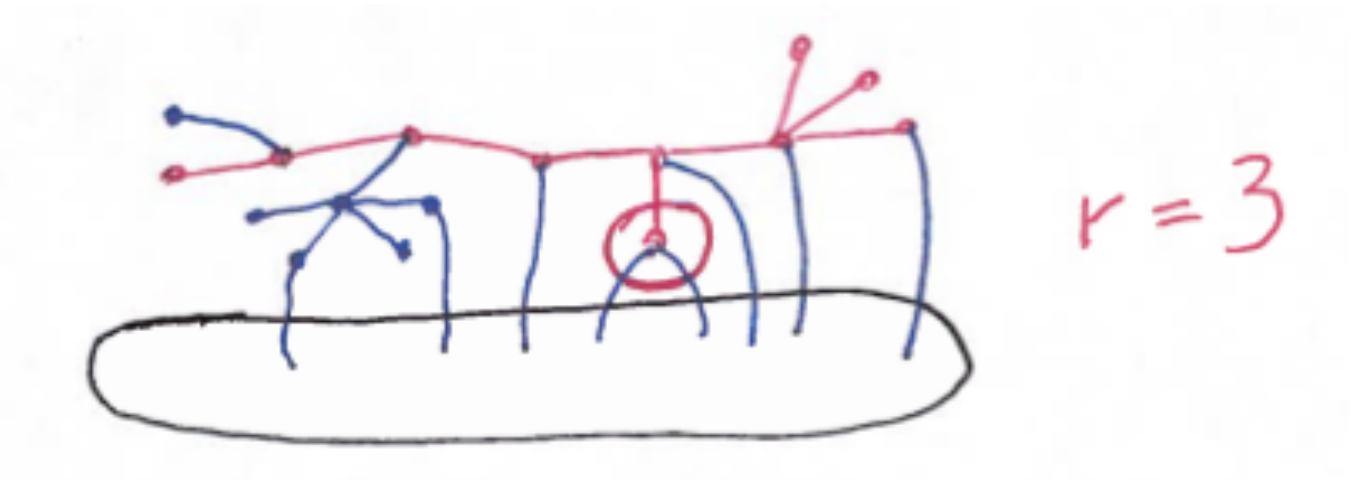
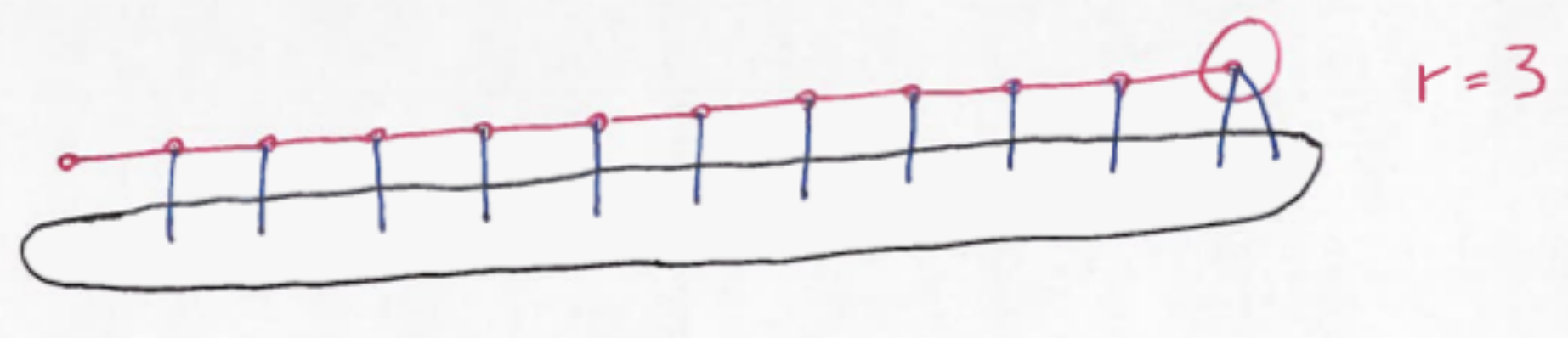


Theorem (Achlioptas and M)

For any $c \neq c^*$ (the r-core threshold), the maximum depth in $H^k(n, M = cn)$ is $O(\log n)$.

Key Step



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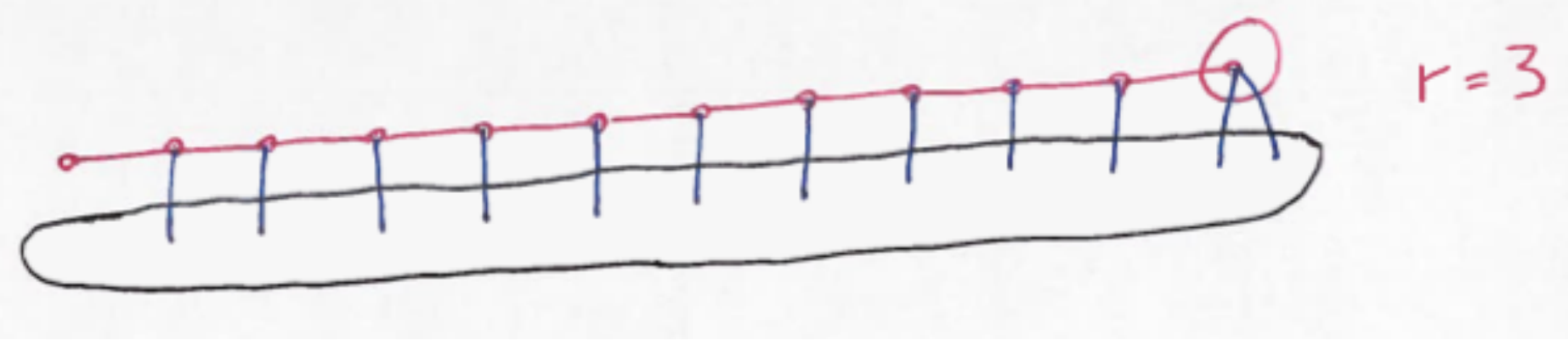

When we remove a vertex, the expected number of degree r neighbours is at most $1 - \epsilon$.

Inside the Clustering Threshold

c^* is the clustering threshold.

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c^* is the clustering threshold.

Theorem (Gao and M)

For sufficiently small $\delta > 0$, and $c = c^* + n^{-\delta}$:

- *Any pair of 2-core solutions must differ on at least $n^{1-\beta}$ variables.*
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($\beta \to 0$ with δ.)

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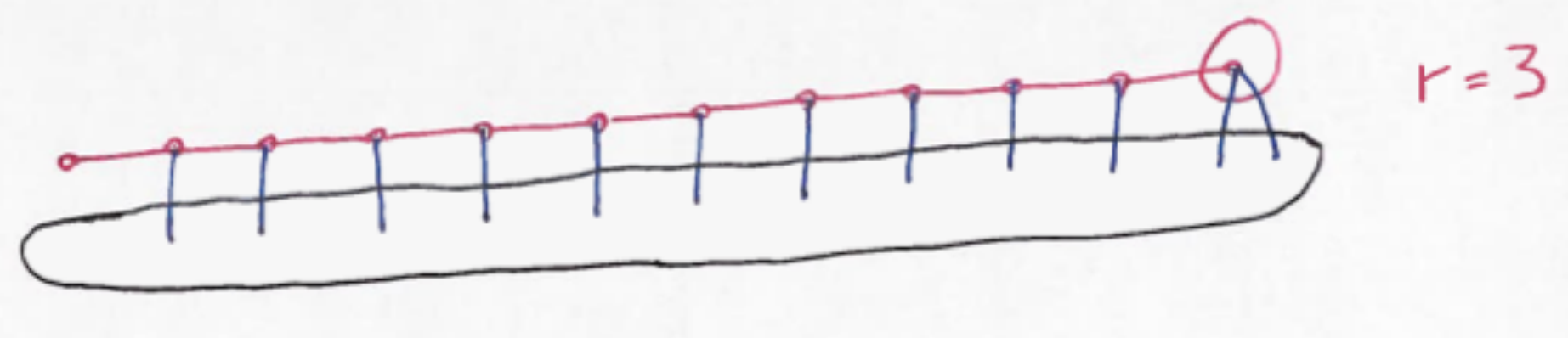
Challenge



When we remove a vertex, the expected number of degree r neighbours is at most $1 - \epsilon$.

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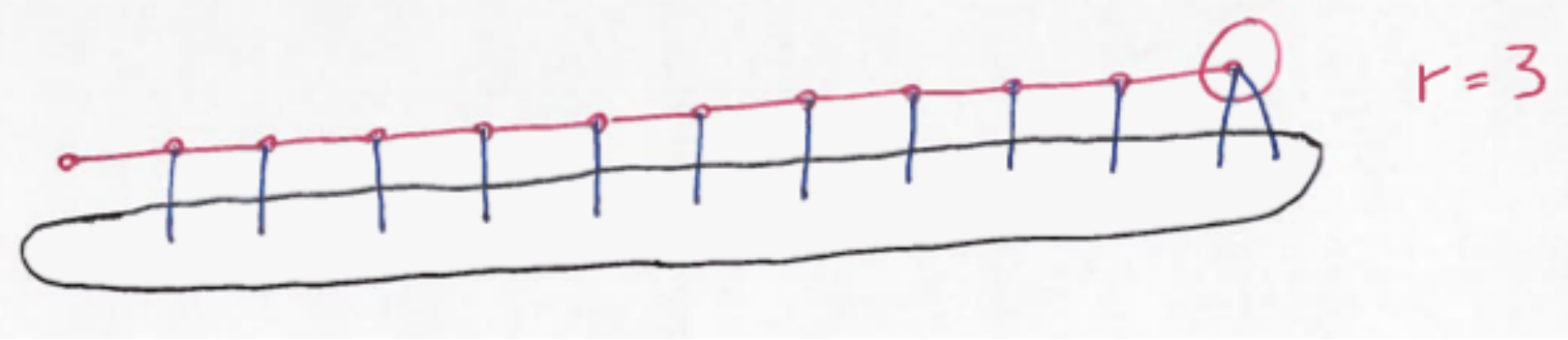
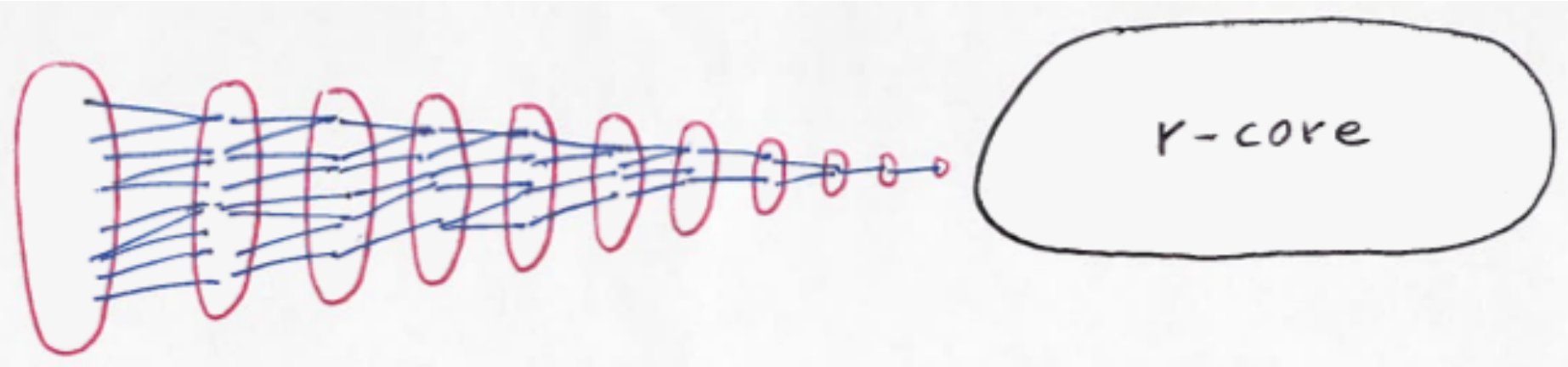


When we remove a vertex, the expected number of degree r neighbours approaches 1, as we approach the r-core.

Parallel Stripping

Parallel Stripping Process: At each iteration, simultaneously remove every vertex of degree less than r.



We begin by determining the number of iterations:

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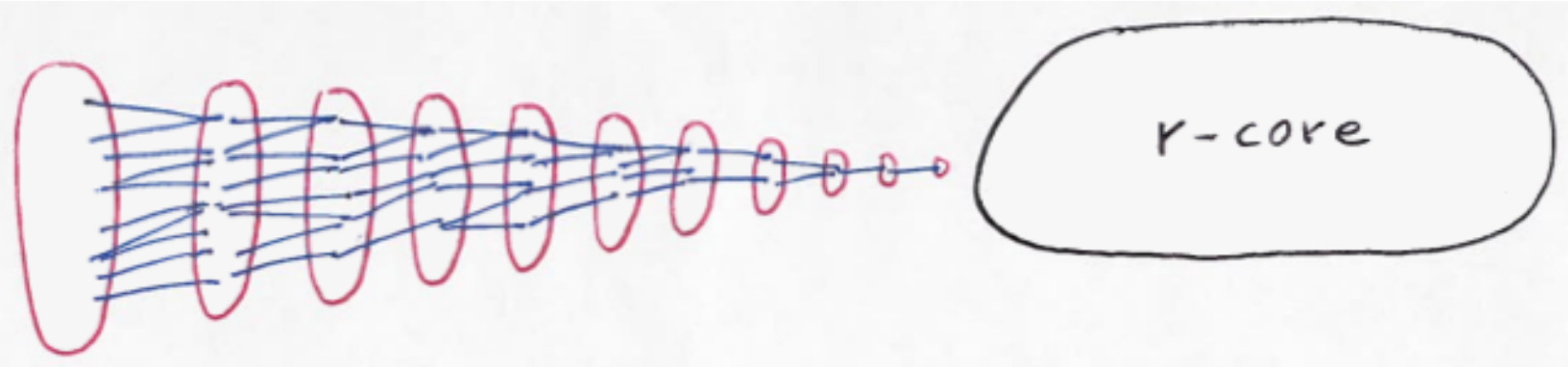

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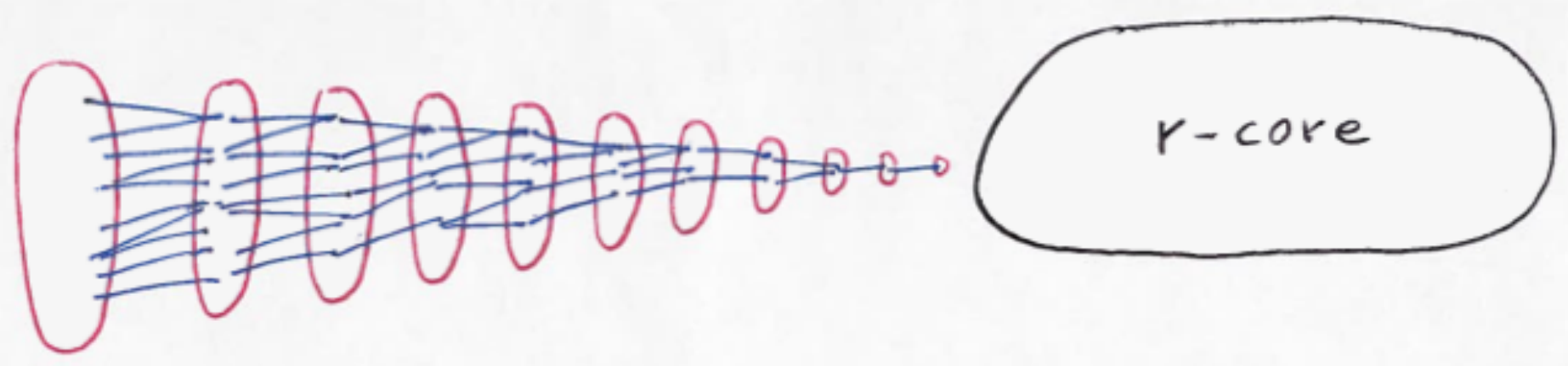
This lower bounds the maximum depth.

Edge Switching

We carry out the parallel stripping process.



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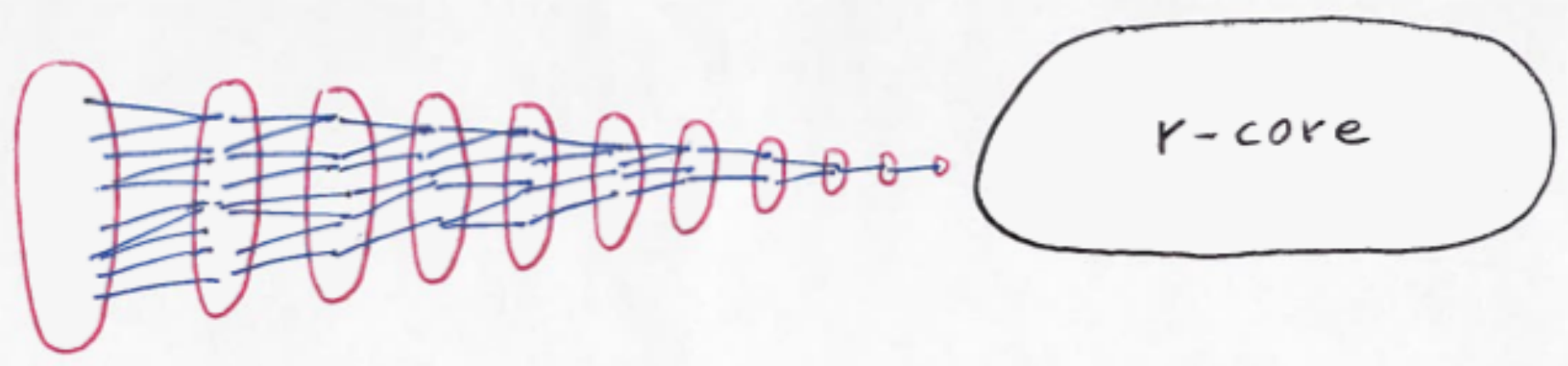

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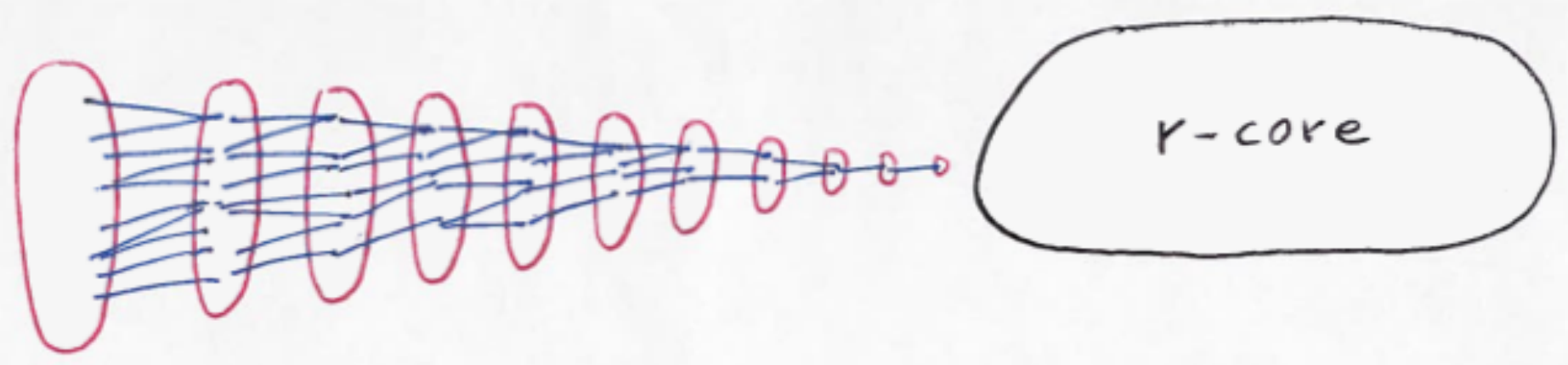

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Equivalently - we first expose the vertices deleted in each iteration, but not the edges between them.

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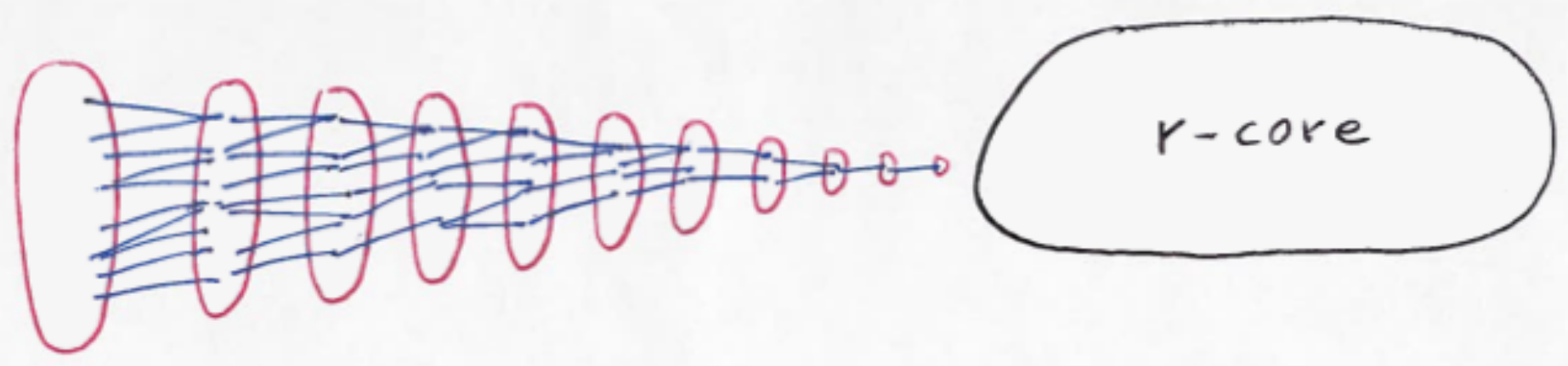

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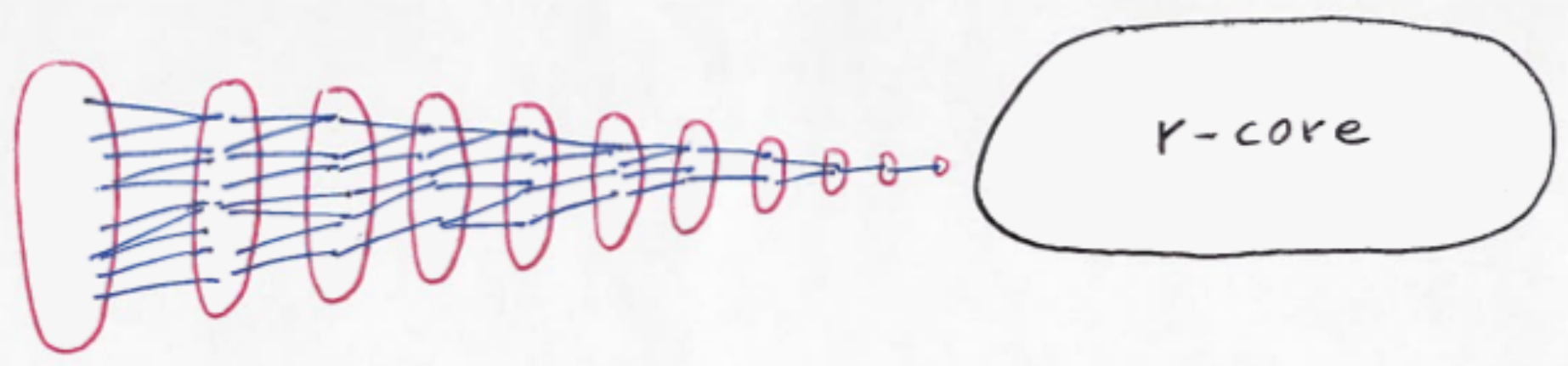

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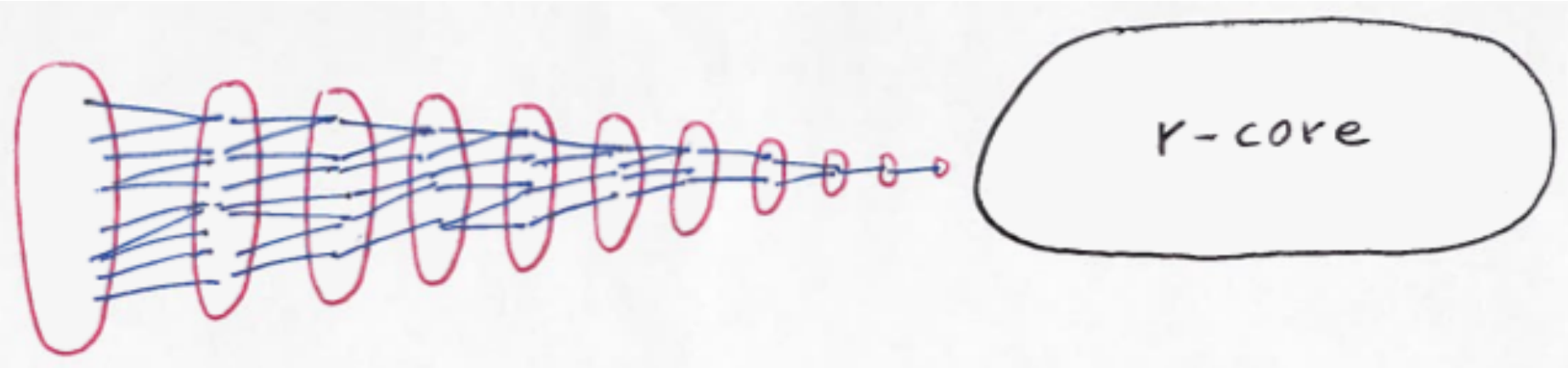
This provides enough randomness for us to analyze the stripping sequence leading to a particular vertex being removed.

Big Steps Required Inside the Clusters

Within each C_σ, changing some variables requires changing $n^{O(\delta)}$ others.



Further Challenges

- Push further into the clustering threshold.
- Gain a better rigorous understanding of clusters for other CSP's.