Requiring Pairwise Nonadjacent Chords in Cycles

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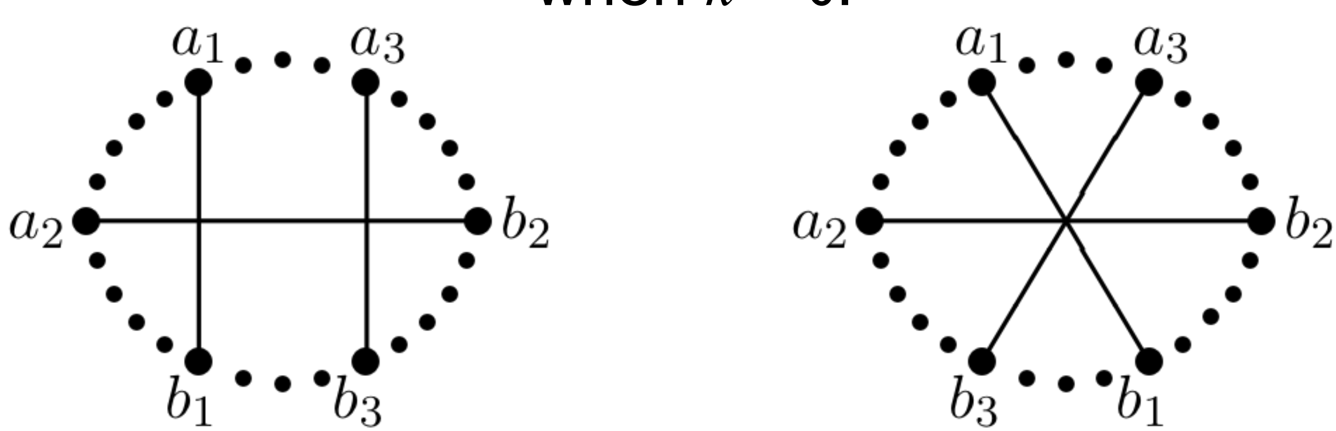
For each $k \geq 4$, let $\mathcal{G}_k$ be the class of all graphs in which every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords.

So, $\mathcal{G}_4$ is the class of chordal graphs.

Theorem: A graph is in $\mathcal{G}_k$ if and only if every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords, *each crossing another*.

when $k = 6$:

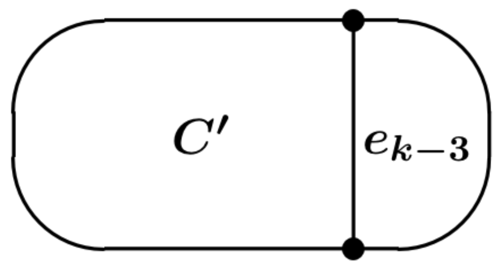


Proof (only if):

Suppose $k \geq 4$ and every cycle of length $\geq k$ has $\geq k-3$ pairwise adjacent chords.

Assume C a minimum-length cycle with $|C| \geq k$ that does not have $k-3$ pairwise adjacent chords *with each crossing another*.

So $k \geq 6$ and C has $k-3 \geq 3$ chords $e_1,\ldots,e_{k-3}$ where (say) e_{k-3} doesn't cross another.



Say $e_1,\ldots,e_h$ are chords of C' and $e_{h+1},\ldots,e_{k\text{-}4}$ are not, where

$$\left\lceil \frac{k-4}{2} \right\rceil \leq h \leq k-4.$$

If $h = k-4$, then $k \leq k + (k-6) = 2(k-4) + 2 \leq |C'| < |C|$ →←

If $h < k-4$, then $k-2 = (k-4) + 2 \leq |C'| < |C|$, so $k \leq |C^*| < |C|$ →←

For each $k \geq 4$, let $\mathcal{G}_k$ be the class of all graphs in which every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords.

So, $\mathcal{G}_4$ is the class of chordal graphs.

And $\mathcal{G}_5$ is the class of distance-hereditary graphs.

[E. Howorka, *QJM-O* 1977]

Corollary: A graph is distance-hereditary if and only if every cycle of length ≥ 5 has *nonadjacent* chords.

For each $k \geq 4$, let $\mathcal{G}_k$ be the class of all graphs in which every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords.

Summary:

A graph is in $\mathcal{G}_4$ iff it is chordal.

A graph is in $\mathcal{G}_5$ iff it is distance-hereditary.

A graph is in $\mathcal{G}_6$ iff every induced hamiltonian subgraph of order ≥ 6 either *contains* a $K_{3,3}$ subgraph or *is* a triangular prism.

A graph is in $\mathcal{G}_7$ iff every induced hamiltonian subgraph of order ≥ 7 is 3-connected and bipartite.

Theorem: When $k \geq 8$, a graph is in $\mathcal{G}_k$ if and only if its circumference is $< k$. (i.e., no cycle has length $\geq k$).

Special Case: $G \in \mathcal{G}_8$ implies no cycle of G has length ≥ 8.

Suppose $G \in \mathcal{G}_8$.
No cycle of G can have length 8 or 9.
So no cycle of G can have length 8, 9, or 10.
So no cycle of G can have length 8, 9, 10, or 11.
So no cycle of G can have length 8, 9, 10, 11 or 12.
So no cycle of G can have length ≥8.

∵ $l = |C| \geq 13$ implies C *cannot* have i-chords whenever $2 \leq i \leq l-7$
yet every i-chord of C *must* have $2 \leq i \leq (l-1)/2 \leq l-7$. $\rightarrow\!\leftarrow$

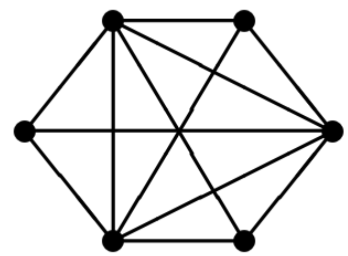
$\Updownarrow$

$l \geq 13$

$\mathcal{G}_4$ is the class of chordal graphs.

$\mathcal{G}_4 \cap \mathcal{G}_5$ is the class of ptolemaic graphs.

Corollary: A graph in $\mathcal{G}_4 \cap \mathcal{G}_5$ is also in $\mathcal{G}_6$ if and only if every induced hamiltonian subgraph of order ≥ 6 *contains* a subgraph $\cong K_6 - K_3$.



$K_6 - K_3$

A graph is in $\mathcal{G}_4$ if and only if, for every k, every k-cycle has at least $k - 3$ chords.

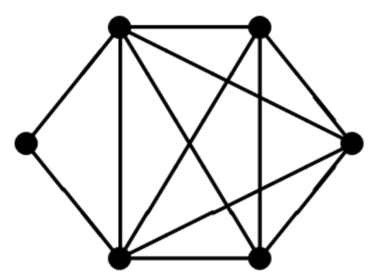
A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5$ if and only if, for every k, every k-cycle has at least $\left\lfloor \frac{3}{2}(k-3) \right\rfloor$ chords.

[E. Howorka, *JGT* 1981]

Theorem: A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ iff, for every k, every k-cycle C has at least

$$f(k) = \frac{1}{2}\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{3}{2}(k-3) \right\rfloor \text{ chords}$$

and $G[V(C)] \not\cong K_6 - K_{1,3}$.



$K_6 - K_{1,3} \notin \mathcal{G}_6$

Theorem: A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ if and only if, for every k, every k-cycle C has at least $f(x) = \frac{1}{2}\left\lfloor\frac{k}{2}\right\rfloor\left\lfloor\frac{3}{2}(k-3)\right\rfloor$ chords *and* $G[V(C)] \ncong K_6 - K_{1,3}$.

Theorem: A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ if and only if every induced hamiltonian subgraph H has at least $\left\lfloor\frac{|V(H)|}{2}\right\rfloor$ universal vertices.

(i.e., "almost most" of its vertices are universal)

Theorem: A graph in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is also in $\mathcal{G}_7$ if and only if its circumference is < 7.

(i.e., no cycle has length ≥ 7).