Requiring Pairwise Nonadjacent Chords in Cycles

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For each $k \geq 4$, let $G_k$ be the class of all graphs in which every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords.

So, $G_4$ is the class of chordal graphs.

**Theorem:** A graph is in $G_k$ if and only if every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords, each crossing another. 

when $k = 6$:
**Proof (only if):**
Suppose \( k \geq 4 \) and every cycle of length \( \geq k \) has \( \geq k - 3 \) pairwise adjacent chords.

Assume \( C \) a minimum-length cycle with \( |C| \geq k \) that does not have \( k - 3 \) pairwise adjacent chords with each crossing another.

So \( k \geq 6 \) and \( C \) has \( k - 3 \geq 3 \) chords \( e_1, \ldots, e_{k-3} \) where (say) \( e_{k-3} \) doesn’t cross another.

Say \( e_1, \ldots, e_h \) are chords of \( C' \) and \( e_{h+1}, \ldots, e_{k-4} \) are not, where
\[
\left\lfloor \frac{k - 4}{2} \right\rfloor \leq h \leq k - 4.
\]

If \( h = k - 4 \), then \( k \leq k + (k - 6) = 2(k - 4) + 2 \leq |C'| < |C| \)

If \( h < k - 4 \), then \( k - 2 = (k - 4) + 2 \leq |C'| < |C| \), so \( k \leq |C^*| < |C| \)
For each \( k \geq 4 \), let \( G_k \) be the class of all graphs in which every cycle of length \( \geq k \) has \( \geq k - 3 \) pairwise nonadjacent chords. So, \( G_4 \) is the class of chordal graphs.

And \( G_5 \) is the class of **distance-hereditary graphs**.

**Corollary:** A graph is distance-hereditary if and only if every cycle of length \( \geq 5 \) has *nonadjacent* chords.

[E. Howorka, QJM-O 1977]
For each $k \geq 4$, let $G_k$ be the class of all graphs in which every cycle of length $\geq k$ has $\geq k - 3$ pairwise nonadjacent chords.

**Summary:**

A graph is in $G_4$ iff it is chordal.

A graph is in $G_5$ iff it is distance-hereditary.

A graph is in $G_6$ iff every induced hamiltonian subgraph of order $\geq 6$ either contains a $K_{3,3}$ subgraph or is a triangular prism.

A graph is in $G_7$ iff every induced hamiltonian subgraph of order $\geq 7$ is 3-connected and bipartite.

**Theorem:** When $k \geq 8$, a graph is in $G_k$ if and only if its circumference is $< k$. (i.e., no cycle has length $\geq k$).
**Special Case:** $G \in G_8$ implies no cycle of $G$ has length $\geq 8$.

Suppose $G \in G_8$.
No cycle of $G$ can have length 8 or 9.
So no cycle of $G$ can have length 8, 9, or 10.
So no cycle of $G$ can have length 8, 9, 10, or 11.
So no cycle of $G$ can have length 8, 9, 10, 11 or 12.
So no cycle of $G$ can have length $\geq 8$.

$\therefore l = |C| \geq 13$ implies $C$ cannot have $i$-chords whenever $2 \leq i \leq l-7$
yet every $i$-chord of $C$ must have $2\leq i \leq (l-1)/2 \leq l-7$.

$\uparrow$

$l \geq 13$
$G_4$ is the class of chordal graphs.

$G_4 \cap G_5$ is the class of ptolemaic graphs.

**Corollary:** A graph in $G_4 \cap G_5$ is also in $G_6$ if and only if every induced hamiltonian subgraph of order $\geq 6$ contains a subgraph $\cong K_6 - K_3$. 

![Graph](image)
A graph is in $G_4$ if and only if, for every $k$,
every $k$-cycle has at least $k - 3$ chords.

A graph is in $G_4 \cap G_5$ if and only if, for every $k$,
every $k$-cycle has at least $\left\lceil \frac{3}{2}(k - 3) \right\rceil$ chords.

[E. Howorka, JGT 1981]

**Theorem:** A graph is in $G_4 \cap G_5 \cap G_6$ iff, for every $k$,
every $k$-cycle $C$ has at least
\[
f(k) = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3}{2}(k - 3) \right\rceil\]
chords
and $G[V(C)] \not\cong K_6 - K_{1,3}$.

$K_6 - K_{1,3} \not\in G_6$
Theorem: A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ if and only if, for every $k$, every $k$-cycle $C$ has at least $f(x) = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{3}{2}(k - 3) \right\rfloor$ chords
and $G[V(C)] \neq K_6 - K_{1,3}$.

Theorem: A graph is in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ if and only if every induced hamiltonian subgraph $H$
has at least $\left\lceil \frac{|V(H)|}{2} \right\rceil$ universal vertices.
(i.e., “almost most” of its vertices are universal)

Theorem: A graph in $\mathcal{G}_4 \cap \mathcal{G}_5 \cap \mathcal{G}_6$ is also in $\mathcal{G}_7$
if and only if its circumference is $< 7$.
(i.e., no cycle has length $\geq 7$).