Combinatorial Algorithms to Solve Network Interdiction and Scheduling Problems with Multiple Parameters

S.T. McCormick; GP Oriolo; B. Peis

Sauder School of Business, UBC; U. Rome; TU Berlin

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Outline

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- We start with an ordinary max flow min cut network with source s, sink t, and capacities c. The capacity of cut S is $\mathrm{cap}_c(S)$.

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- Some thought shows that it is always optimal to destroy arcs belonging to some cut S (that may depend on B).
- Further thought reveals that we should destroy arcs of S greedily, from the max value of $\rho_e = c_e/r_e$ down to the minimum value: "bang for the buck".

The interdiction curve for a fixed cut S

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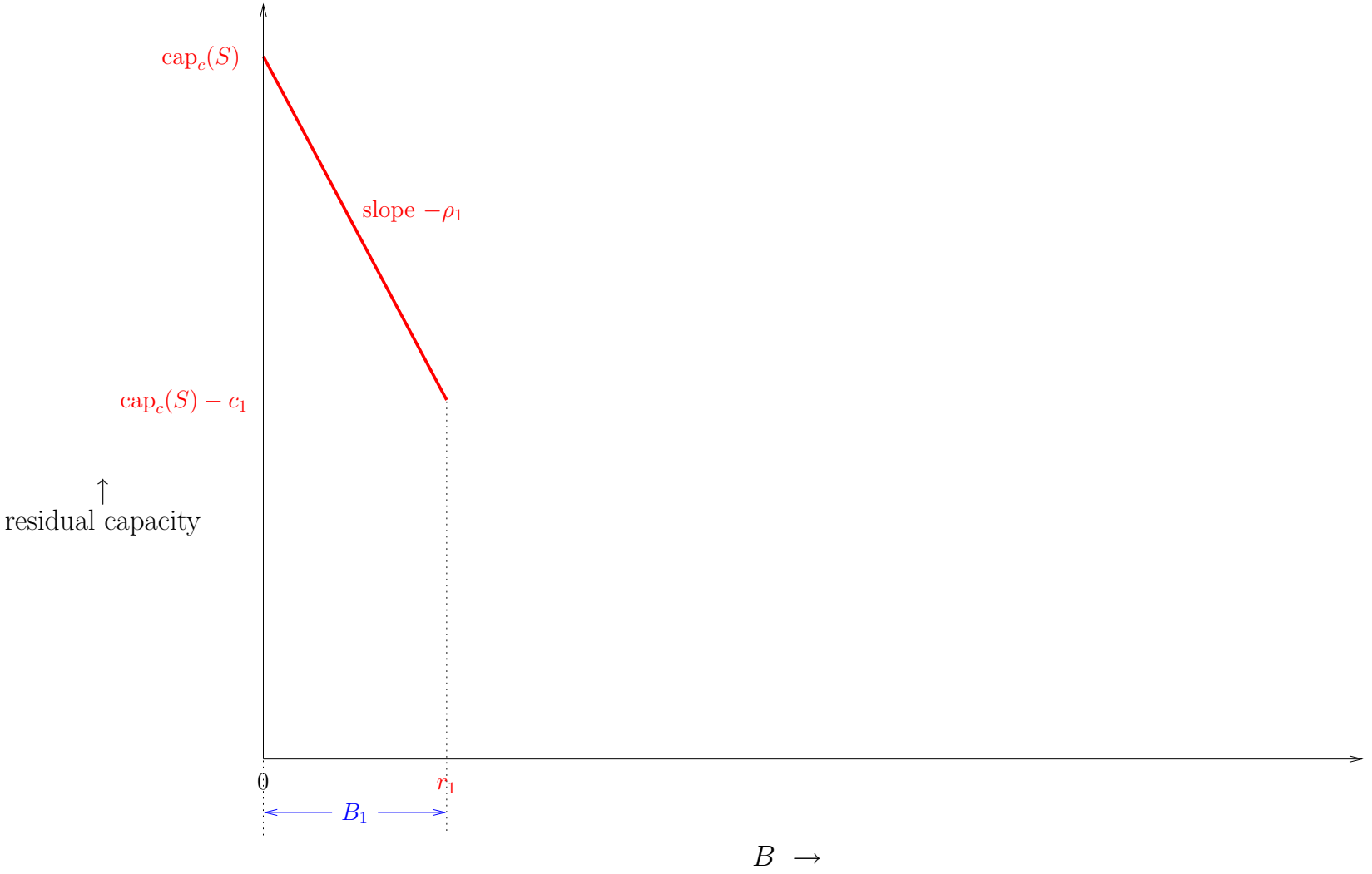

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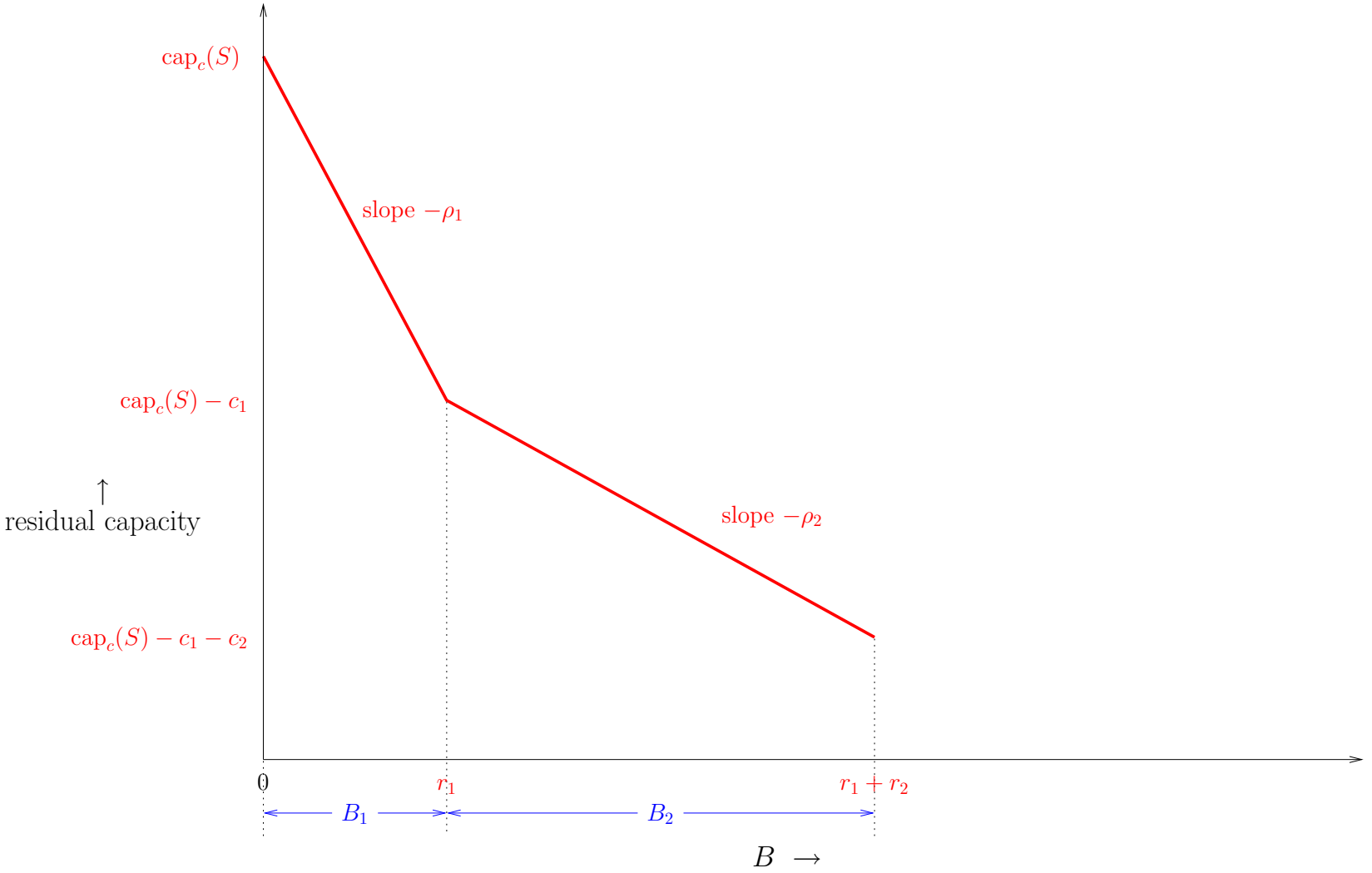

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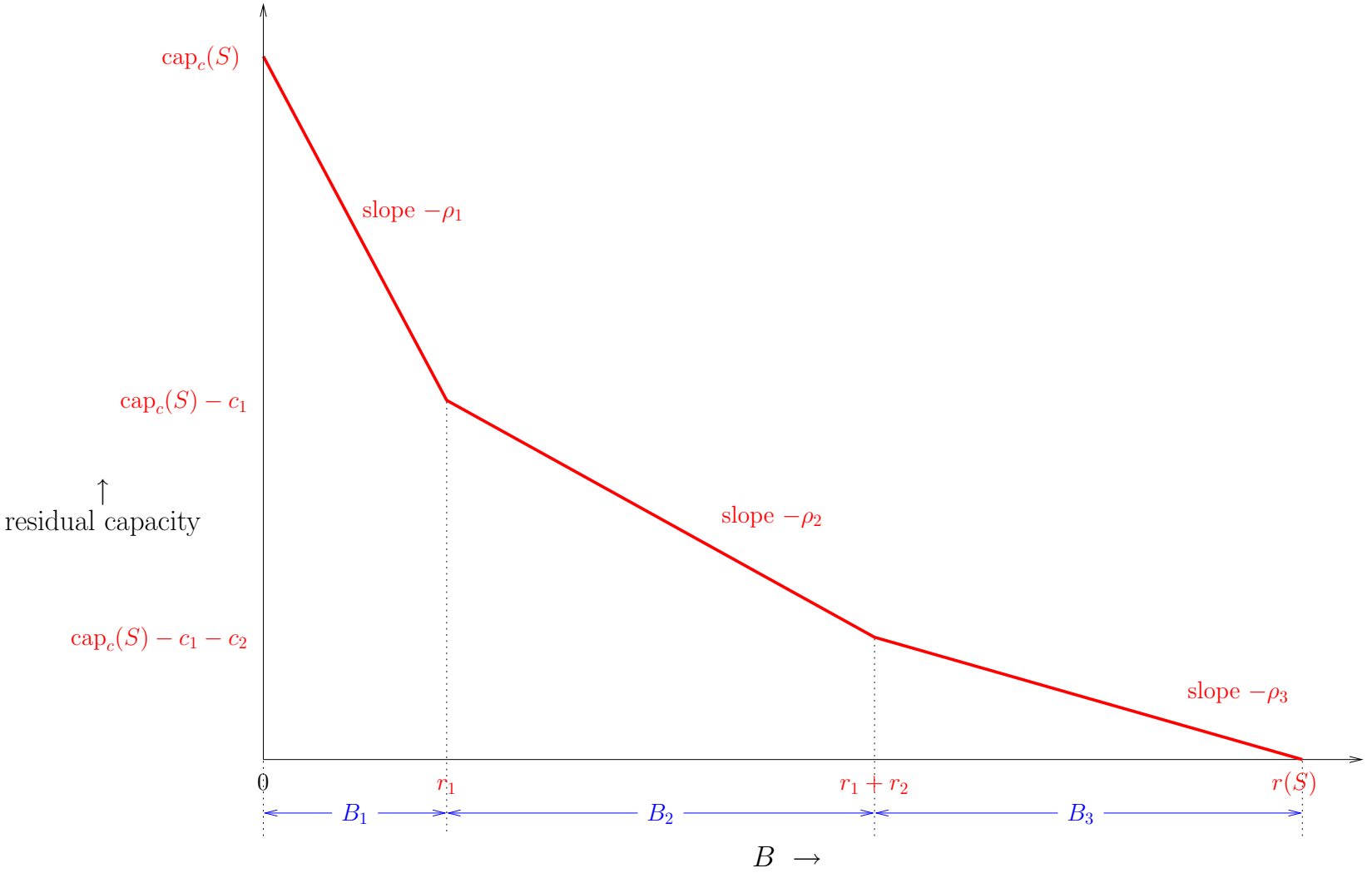
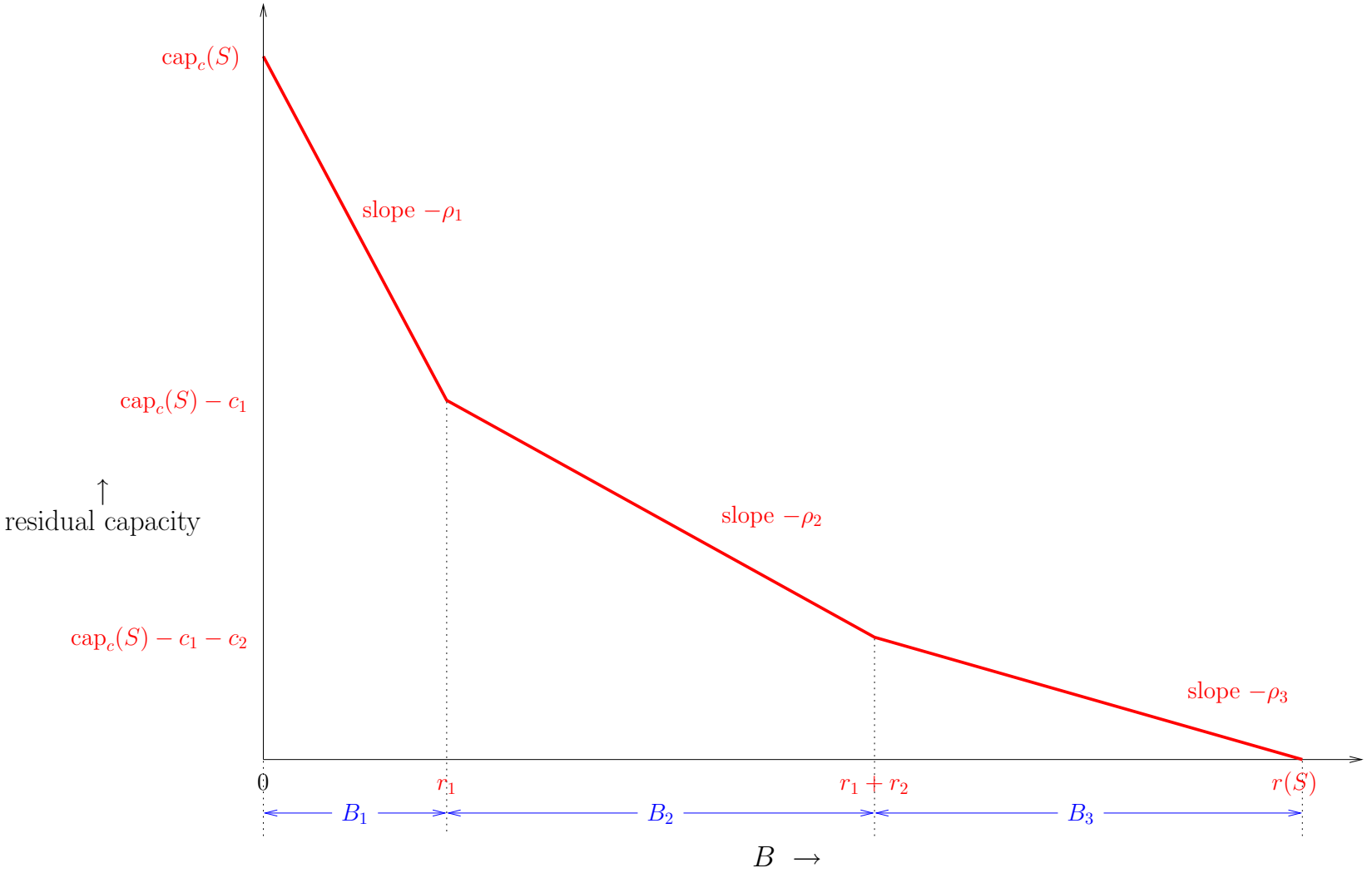


The interdiction curve for a fixed cut S

This curve is piecewise linear convex.



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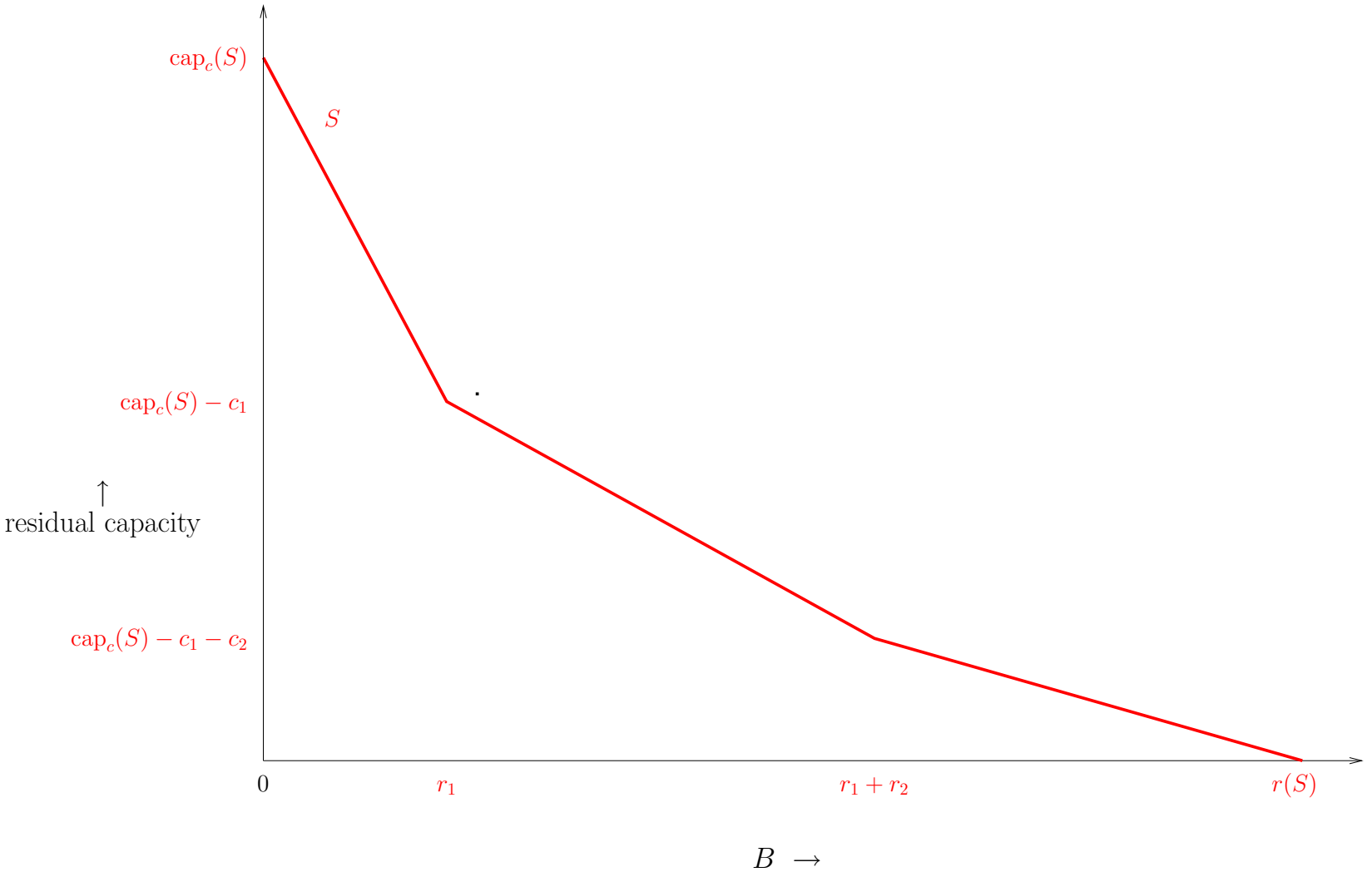

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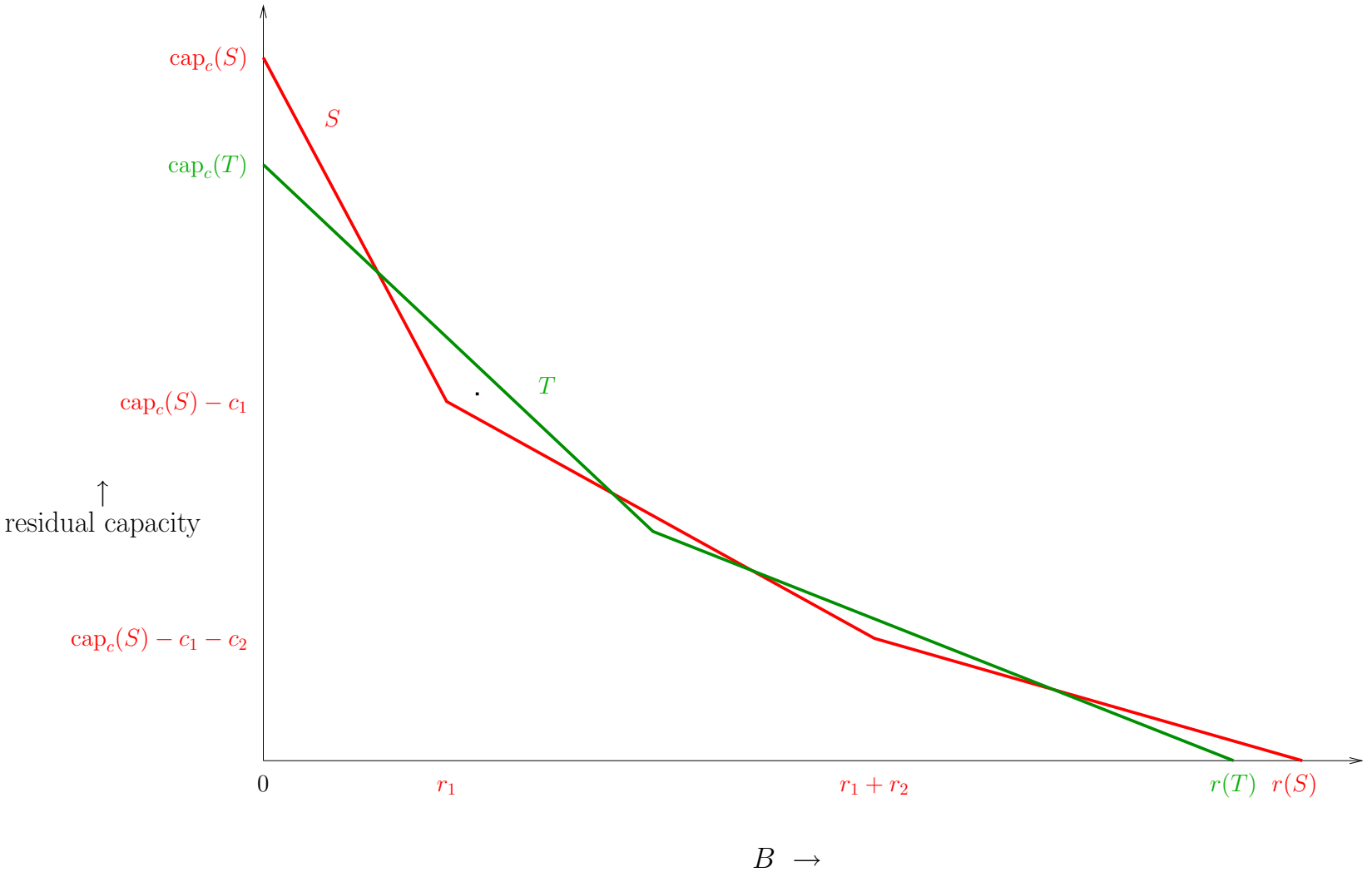

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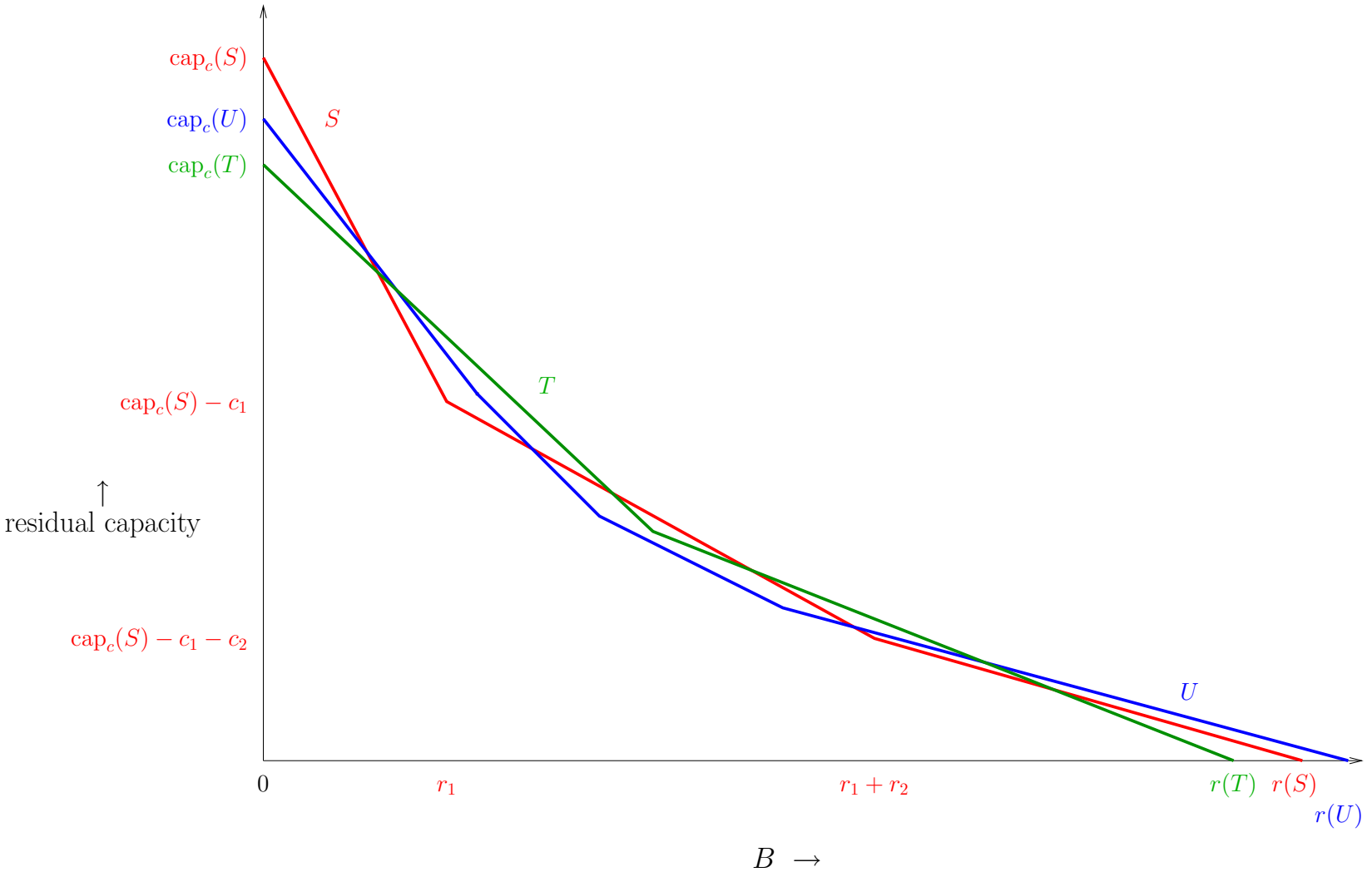

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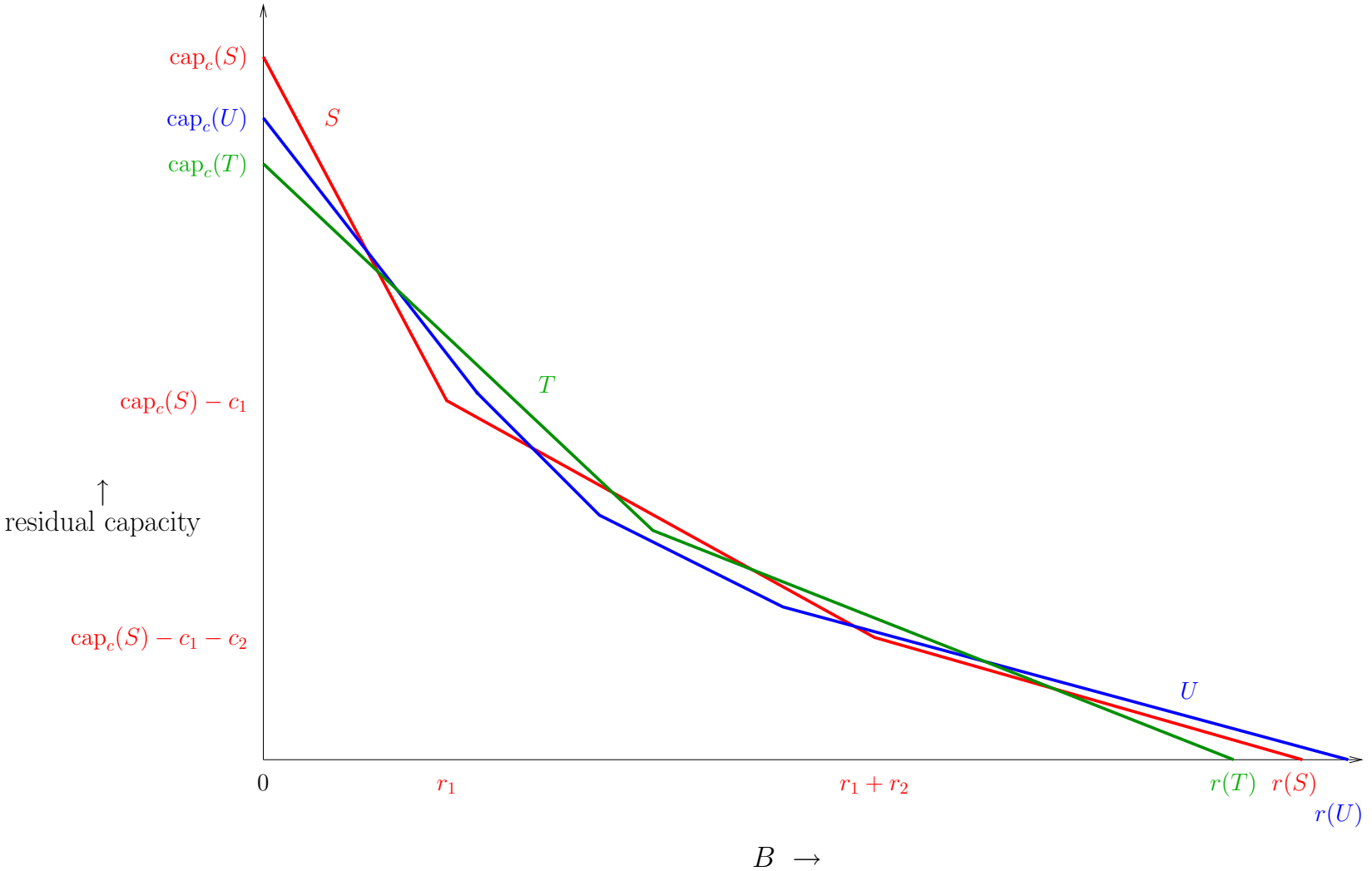
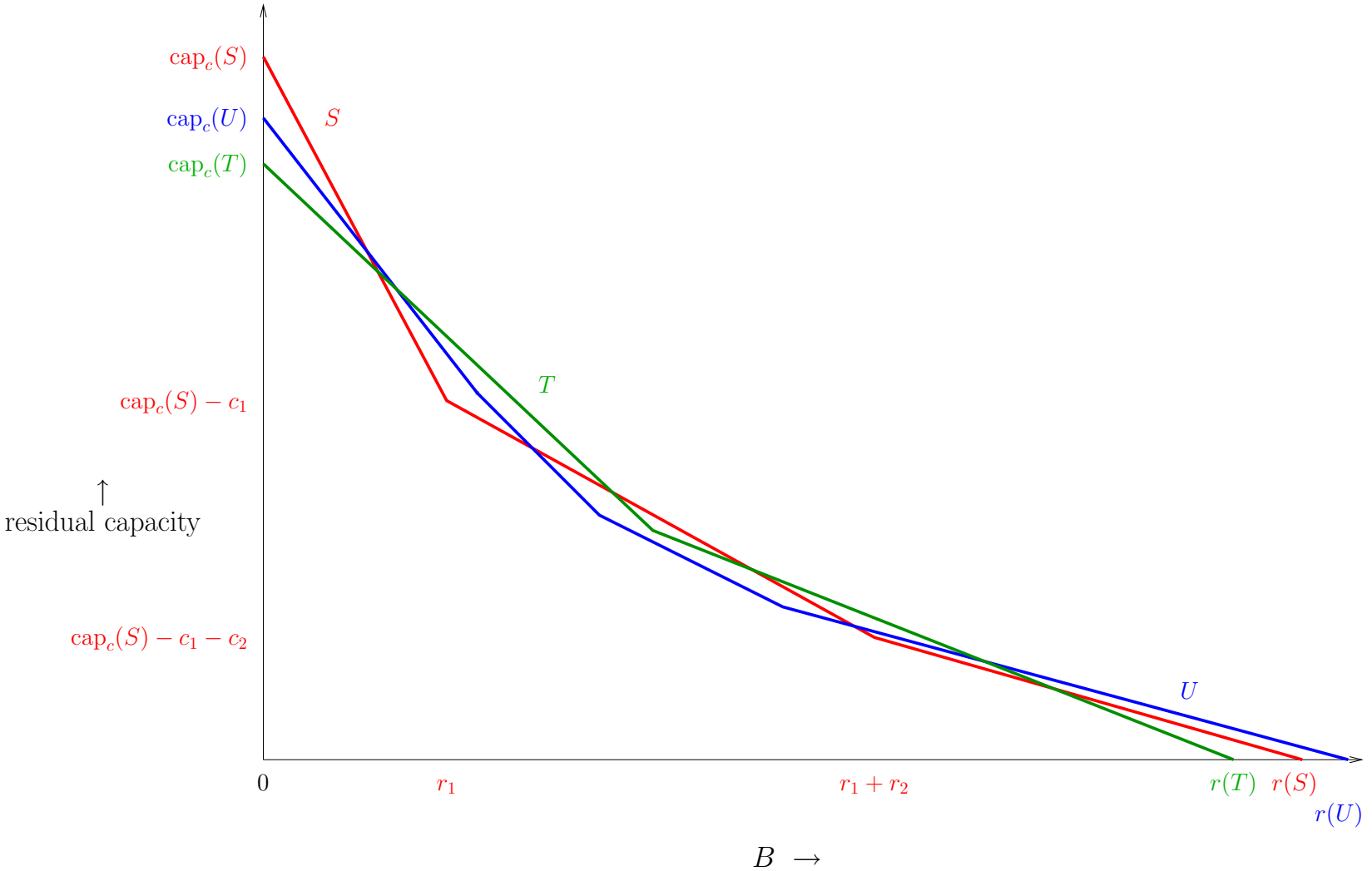
For a given value of B, we just select which S gives the minimum value at B, so the overall curve is the minimum of all the cut-wise curves.



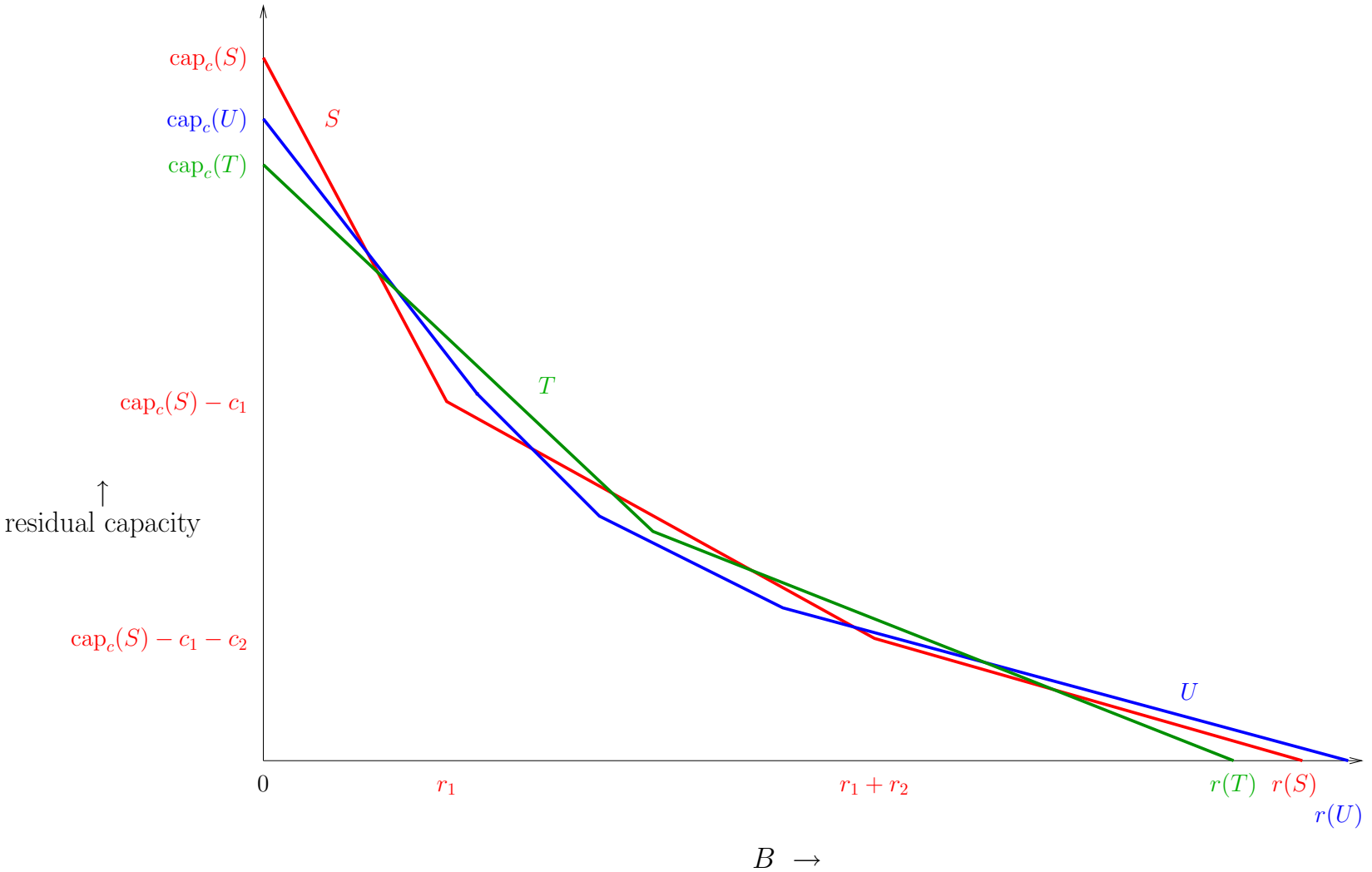
The overall interdiction curve

Unfortunately, the minimum of a bunch of convex curves is not in general convex.



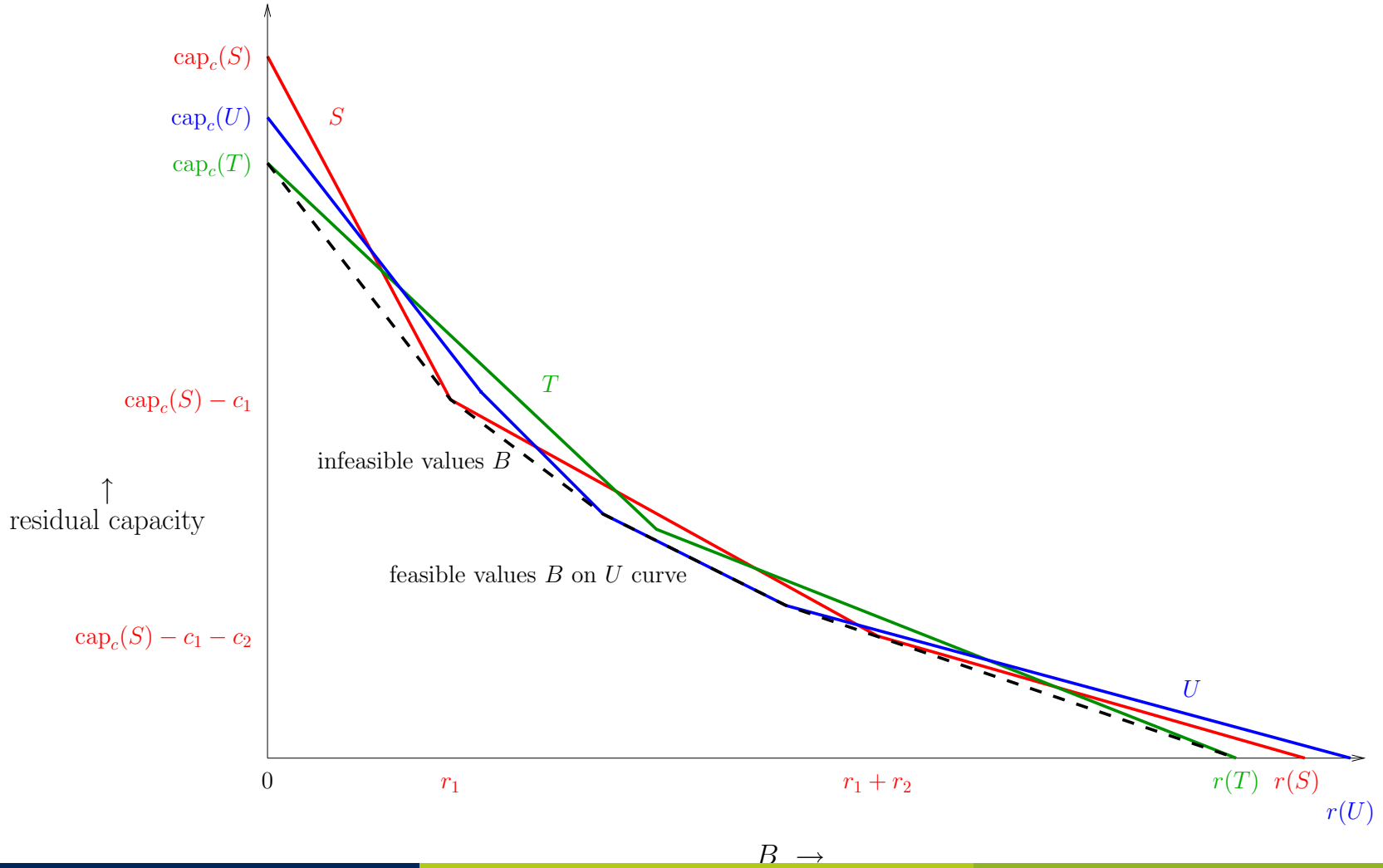
The overall interdiction curve

This is why Network Interdiction is NP Hard (Phillips '93; Wood '93).



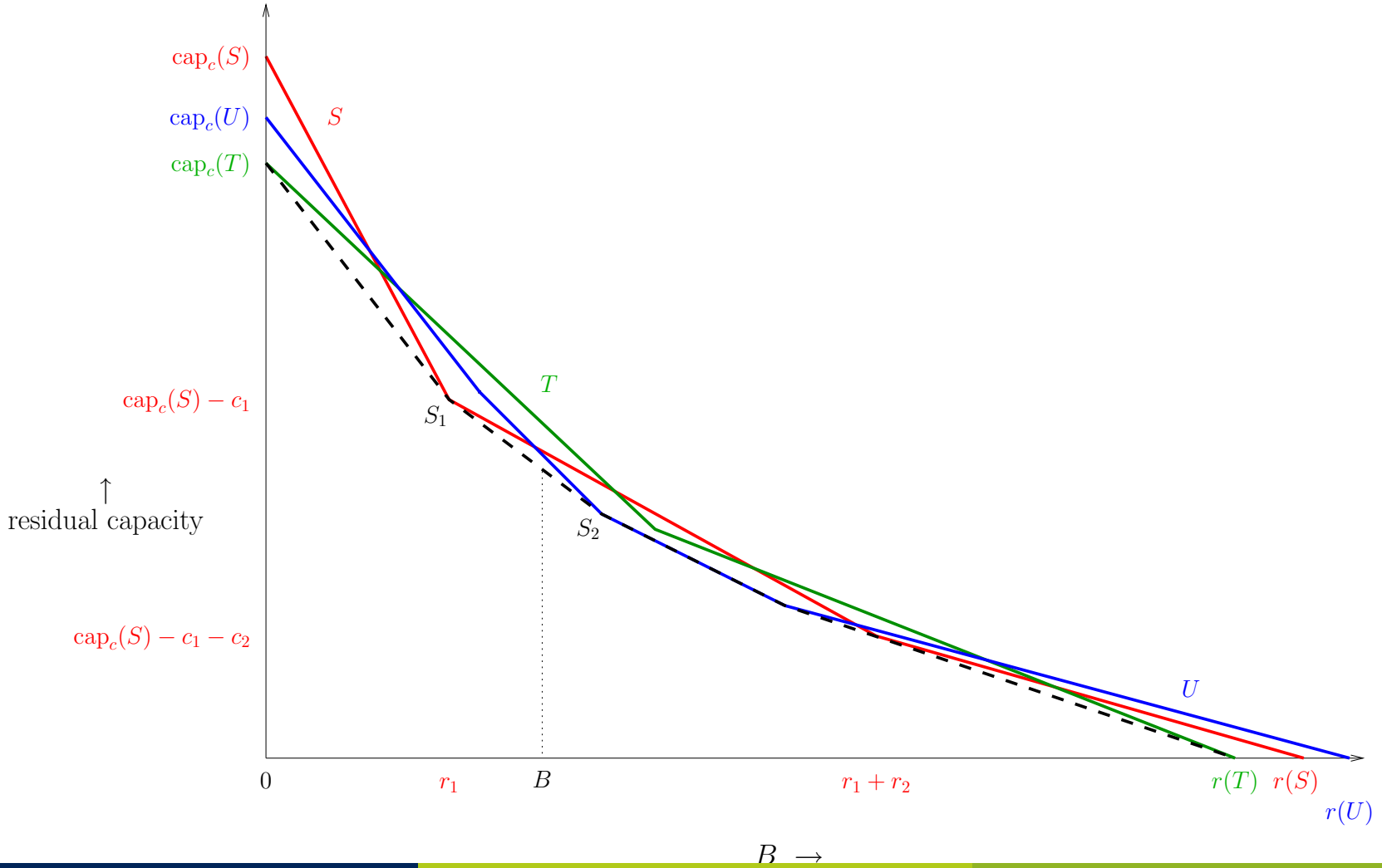
Linearizing the overall curve: the B-profile

If we take the lower envelope, or convex hull, of the overall interdiction curve, we get something tractable, the B-profile.



Linearizing the overall curve: the B-profile

Now budget B corresponds to a convex combination of points coming from the interdiction curves of (one or) two cuts, S_1 and S_2.



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- The algorithmic question is then: Given B, how do we find S_1 and S_2?
- Burch et al write a linear program that can do it, but here we want a combinatorial algorithm to do it.

Outline

The linear program and its dual

- The normal min cut dual LP is

$$\begin{array}{lrcl} \min \sum_{u \to v} c_{uv} y_{uv} & & & \\ \text{s.t. } d_u - d_v + y_{uv} & \geq & 0 & \text{for } u \to v \neq t \to s, \\ d_t - d_s + y_{ts} & \geq & 1 & \\ y_{uv} & \geq & 0 & \text{all } u \to v. \end{array}$$

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- The new LP is then

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The linear program and its dual

- Repeat the new LP with dual variables:

$$
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x_{ts}: & & d_t - d_s + y_{ts} & \geq & 1 & \\
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- When we "primalize" this interdiction dual LP we get new primal variable λ corresponding to the dual constraint $\sum_{u\to v} r_{uv} z_{uv} \leq B$, and the z_{uv}'s give us a second set of capacities.

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- The primal interdiction LP is

$$\begin{array}{rrcl} & \max_{x,\lambda} (x_{ts} - B\lambda) & & \\ d: & \text{s.t. conservation} & & \\ y_{uv}: & 0 \leq x_{uv} & \leq & c_{uv} \\ z_{uv}: & x_{uv} - r_{uv}\lambda & \leq & 0. \end{array}$$

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- Repeat the primal interdiction LP and highlight the two capacities:

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$$x_{uv} \leq \min(c_{uv}, \lambda r_{uv}),$$

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  a *parametric capacity* in the scalar parameter λ.
- So let's investigate the behavior of this parametric min cut problem.

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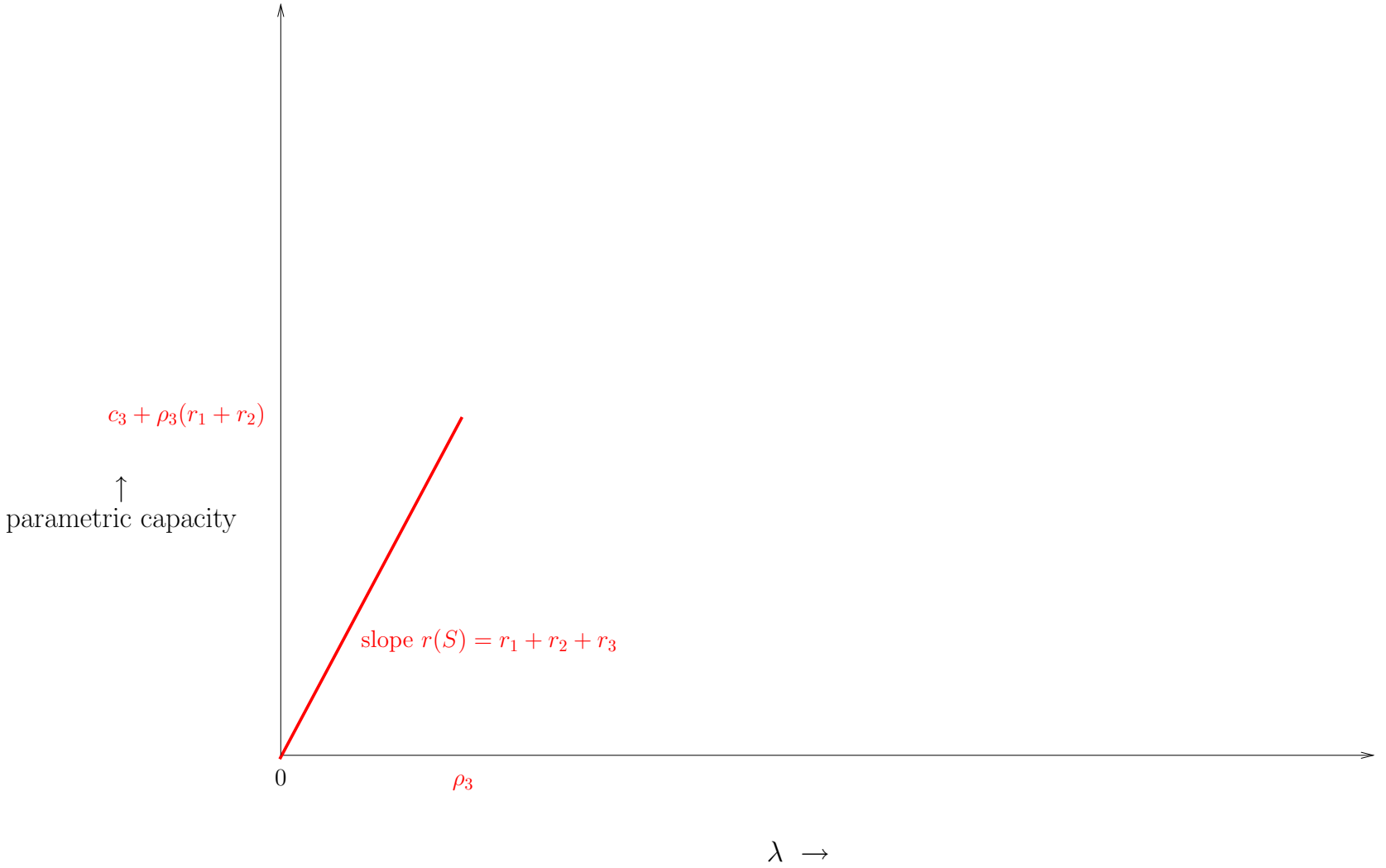
Parametric capacity of fixed cut S

When λ is small, $\text{cap}(S, \lambda) = \lambda \text{cap}_r(S)$.



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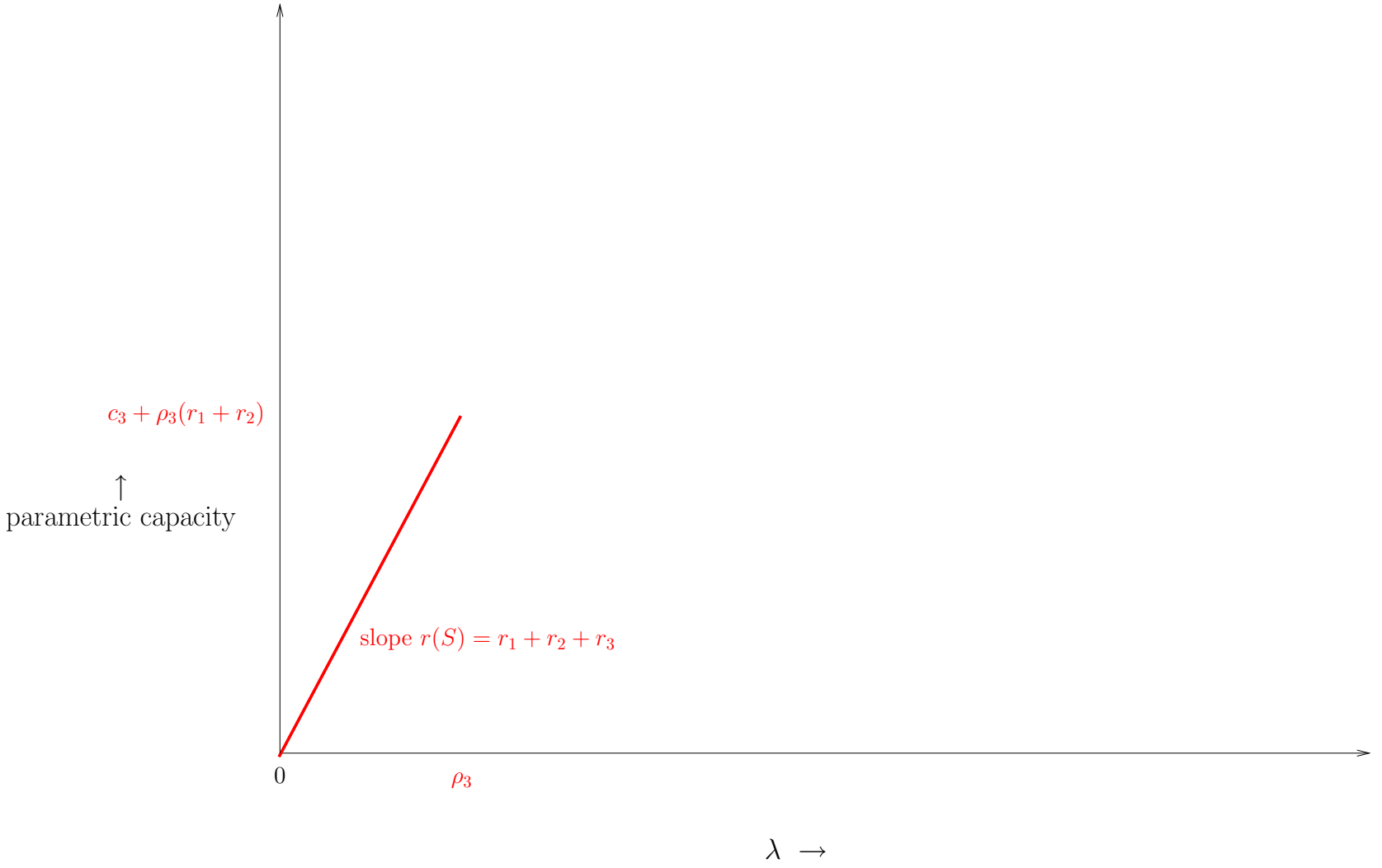
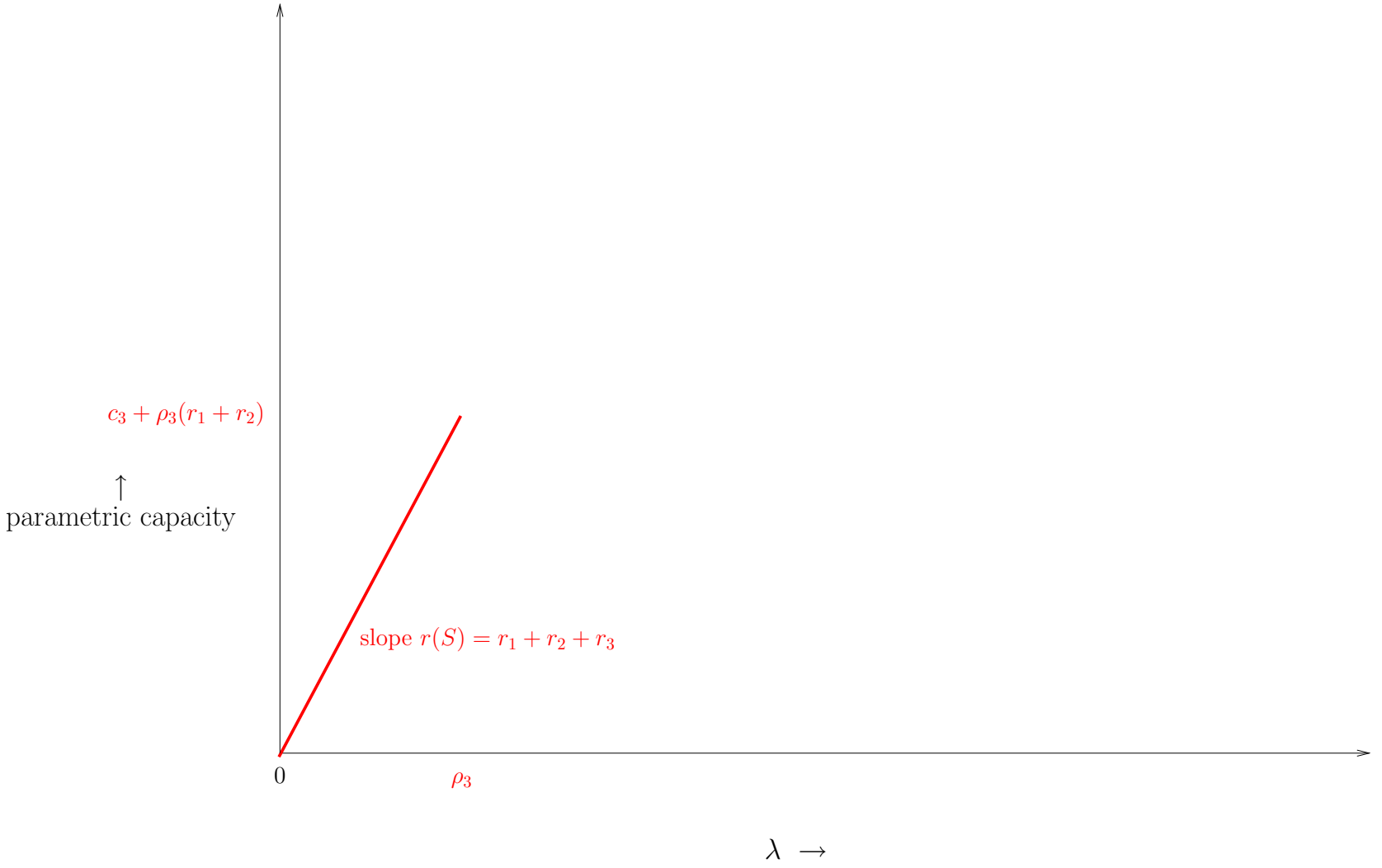
This continues as long as $\lambda r_{uv} \leq c_{uv}$ for all $u \to v \in \delta^+(S)$, or $\lambda \leq \rho_{uv}$.



Parametric capacity of fixed cut S

Thus the first breakpoint is when λ hits $\min_{\delta^+(S)} \rho_{uv}$.



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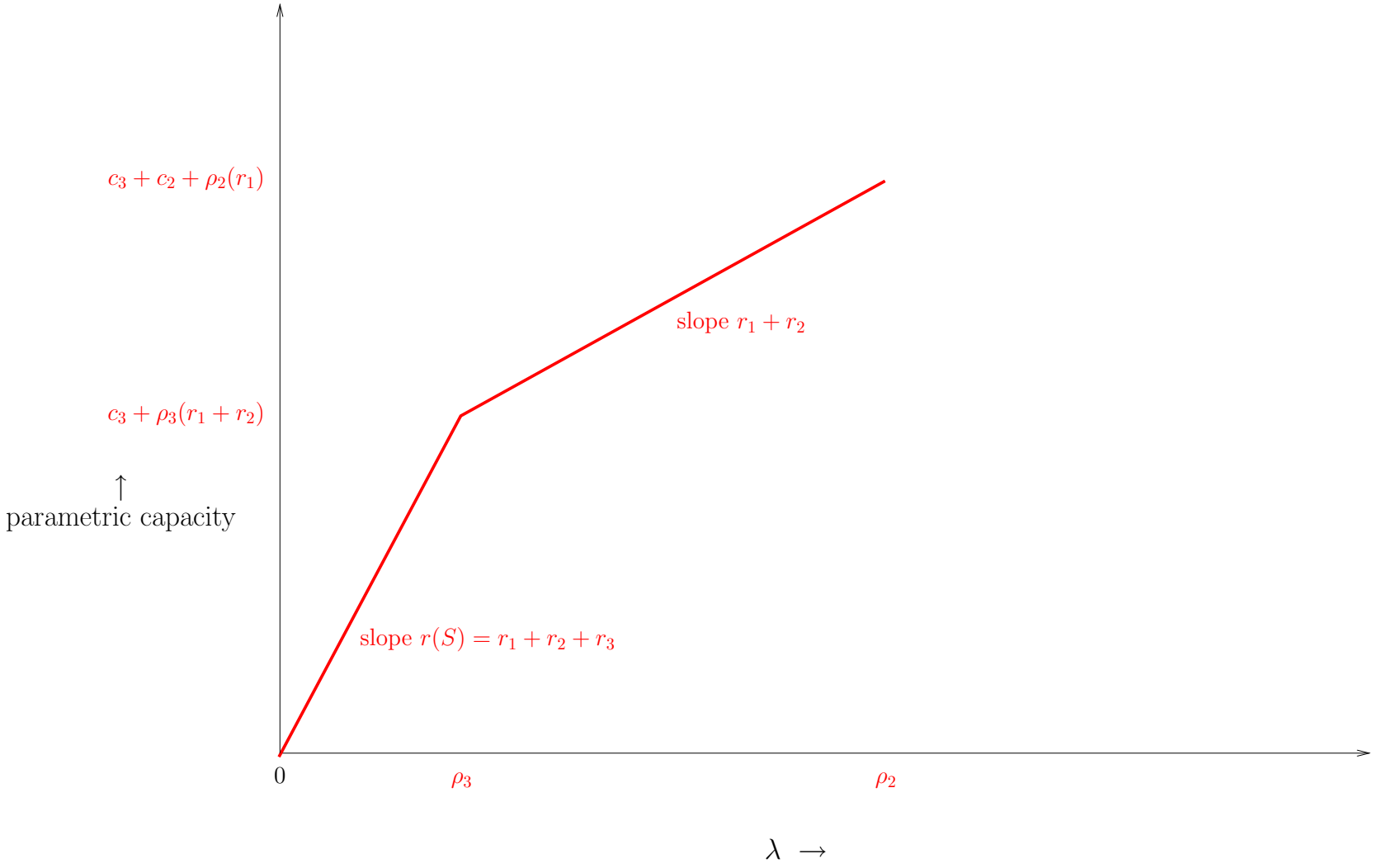

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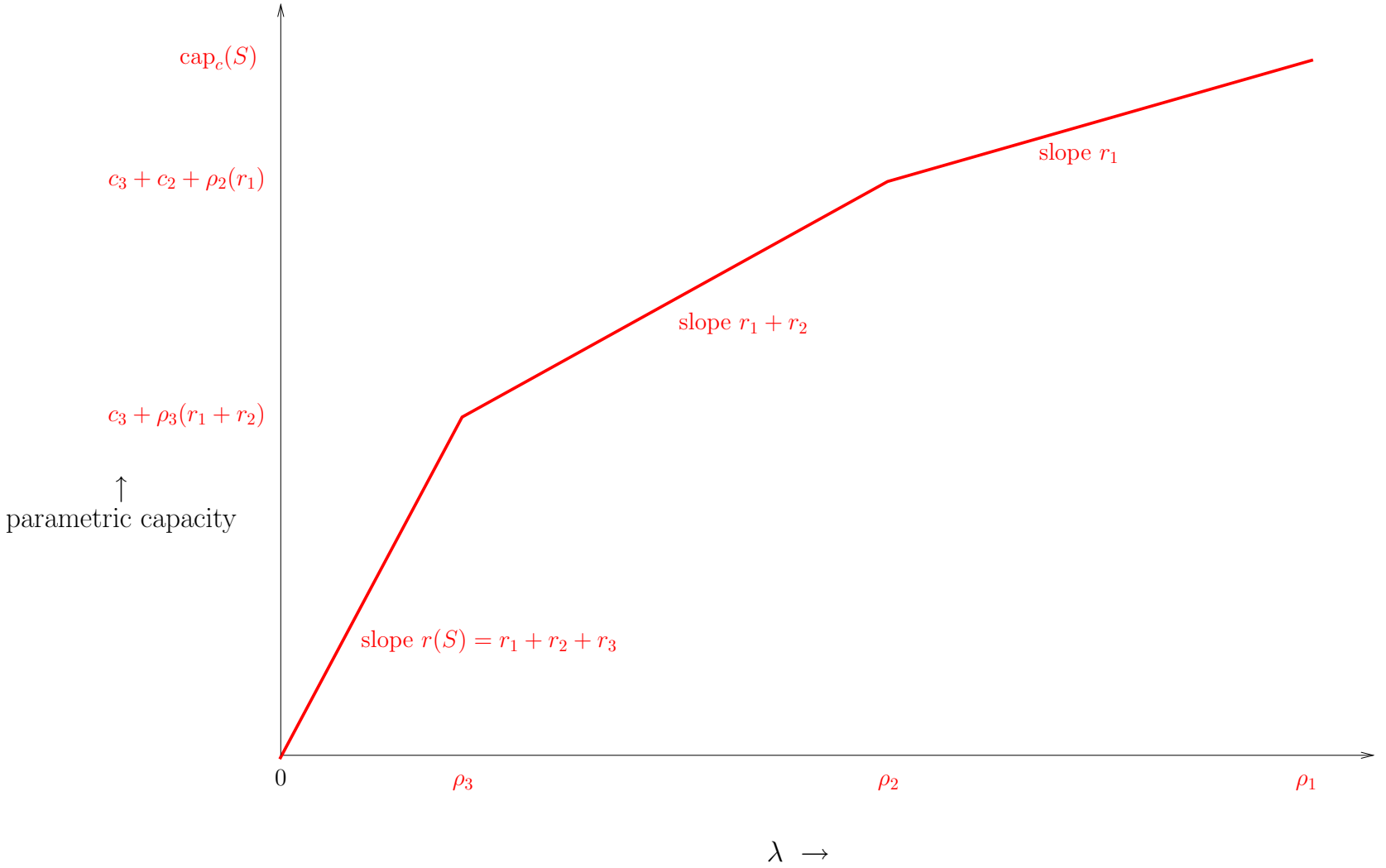

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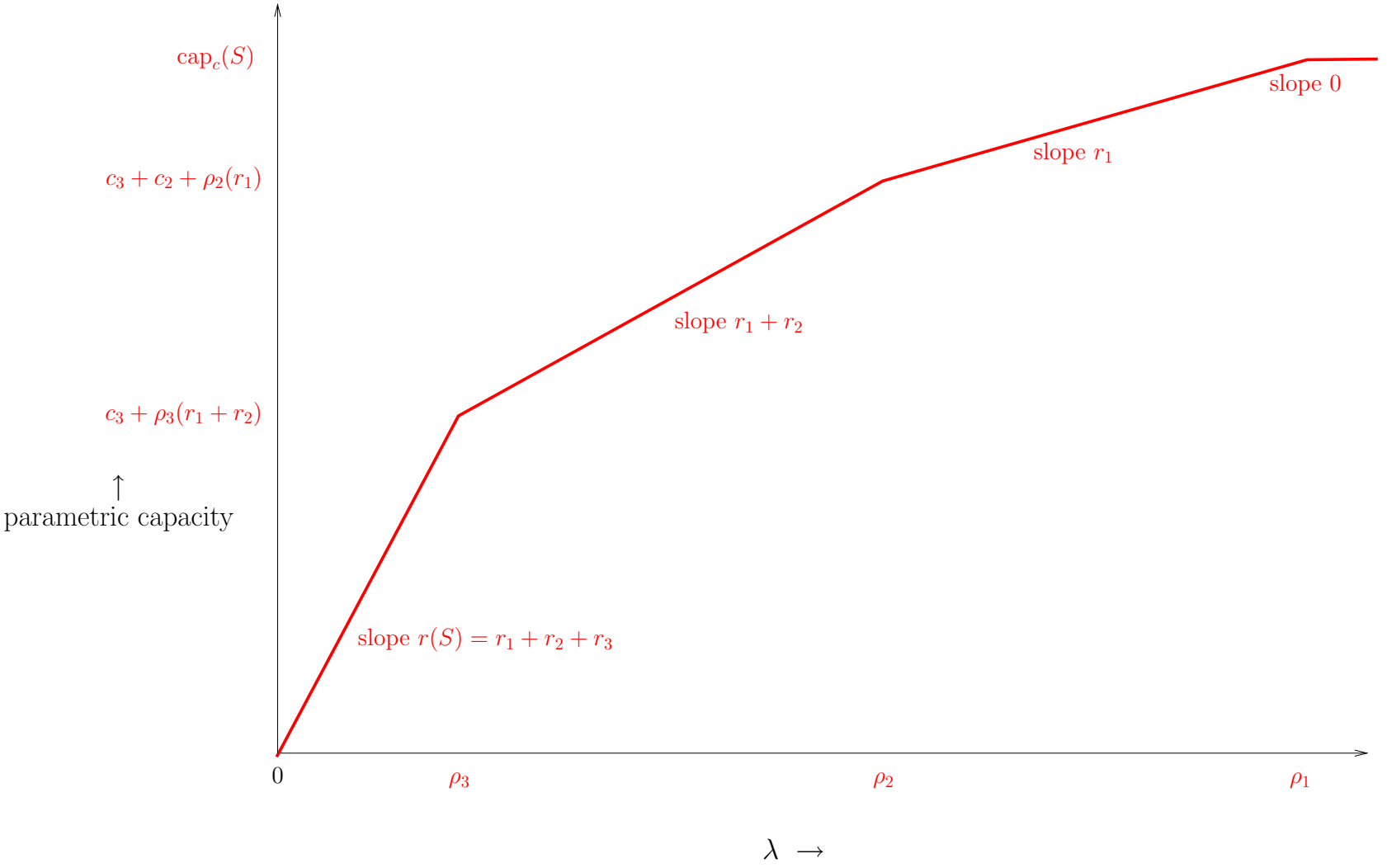
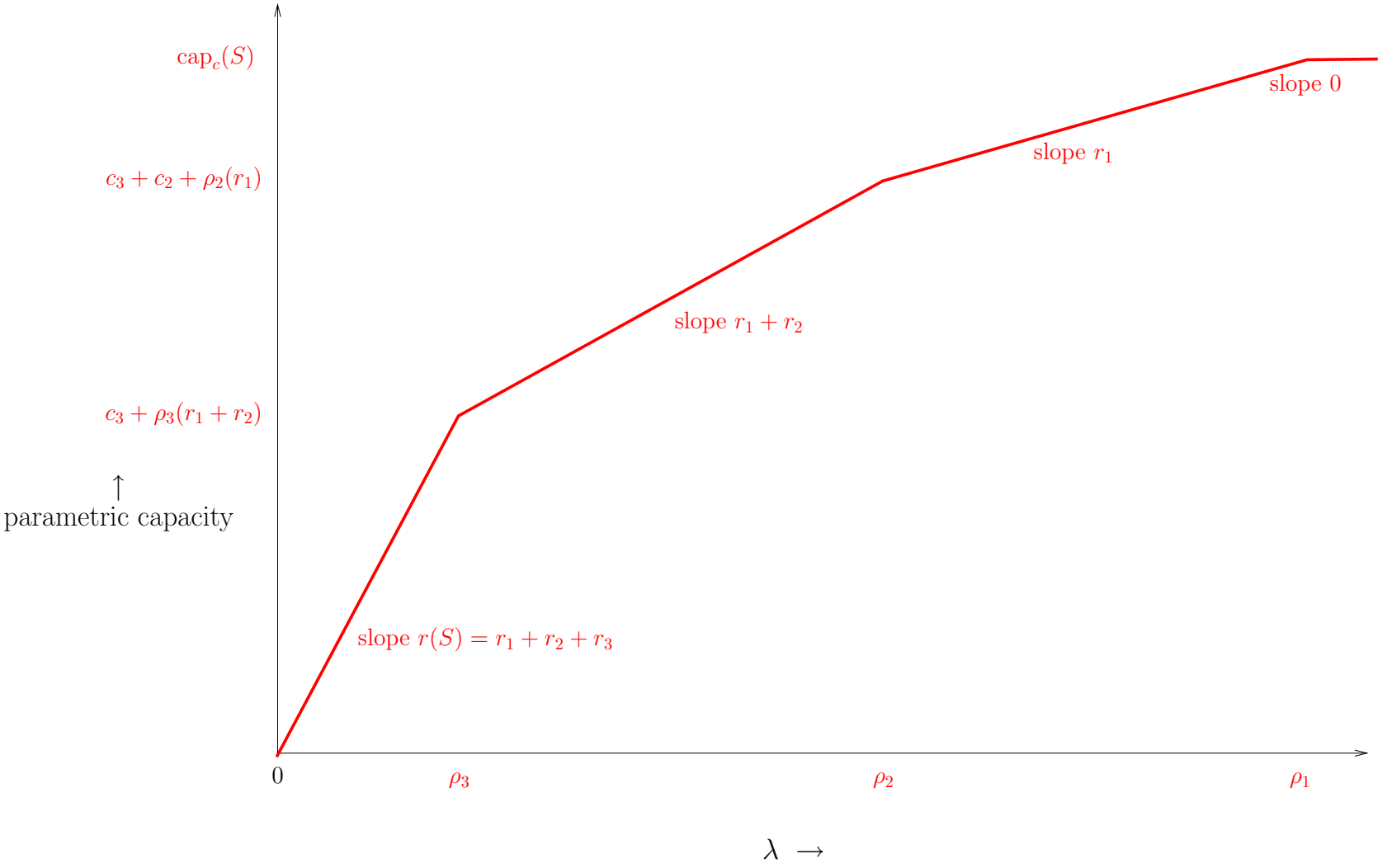


Parametric capacity of fixed cut S

The parametric capacity curve for S is piecewise linear concave.



Conjugate duality between interdiction and parametric capacity for S

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- Thus breakpoints and slopes are interchanged between S's interdiction curve and its parametric capacity curve, though in reverse order and modulo a minus sign.
- In the language of conjugate duality, this is equivalent to saying that the parametric capacity curve $\mathrm{cap}(S, \lambda)$ is the negative of the conjugate dual of the interdiction curve for S, evaluated at $-\lambda$.

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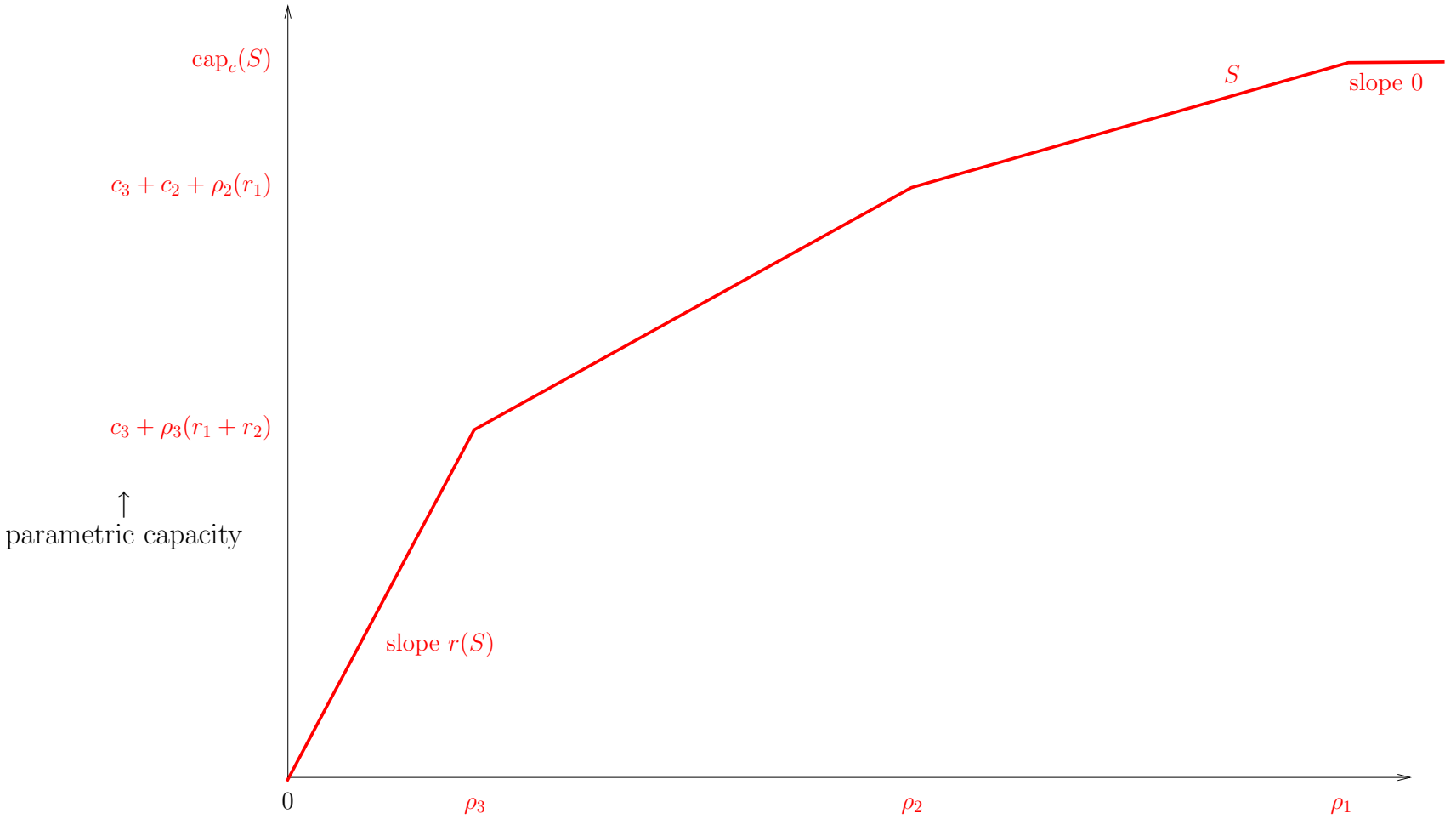

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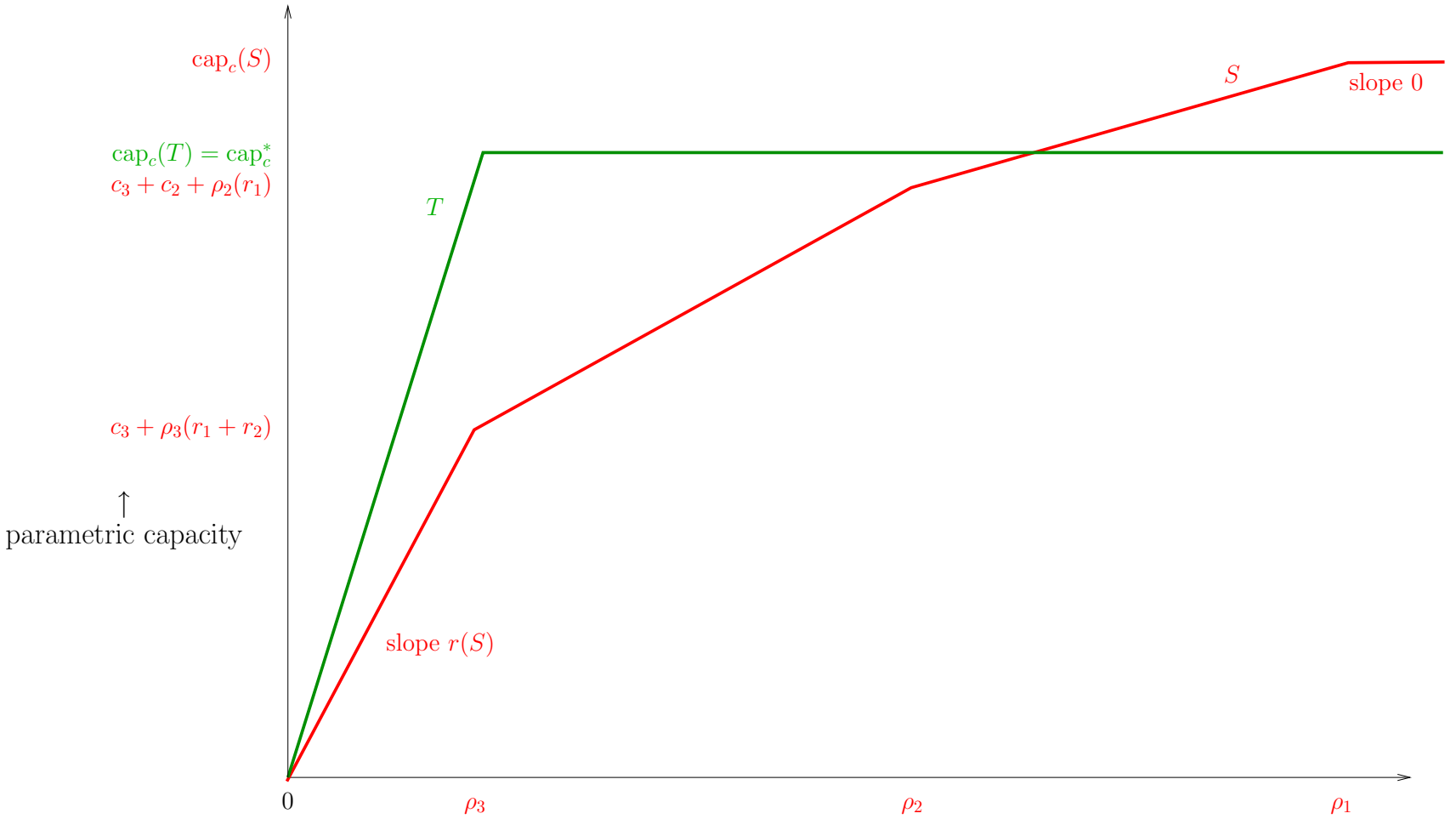

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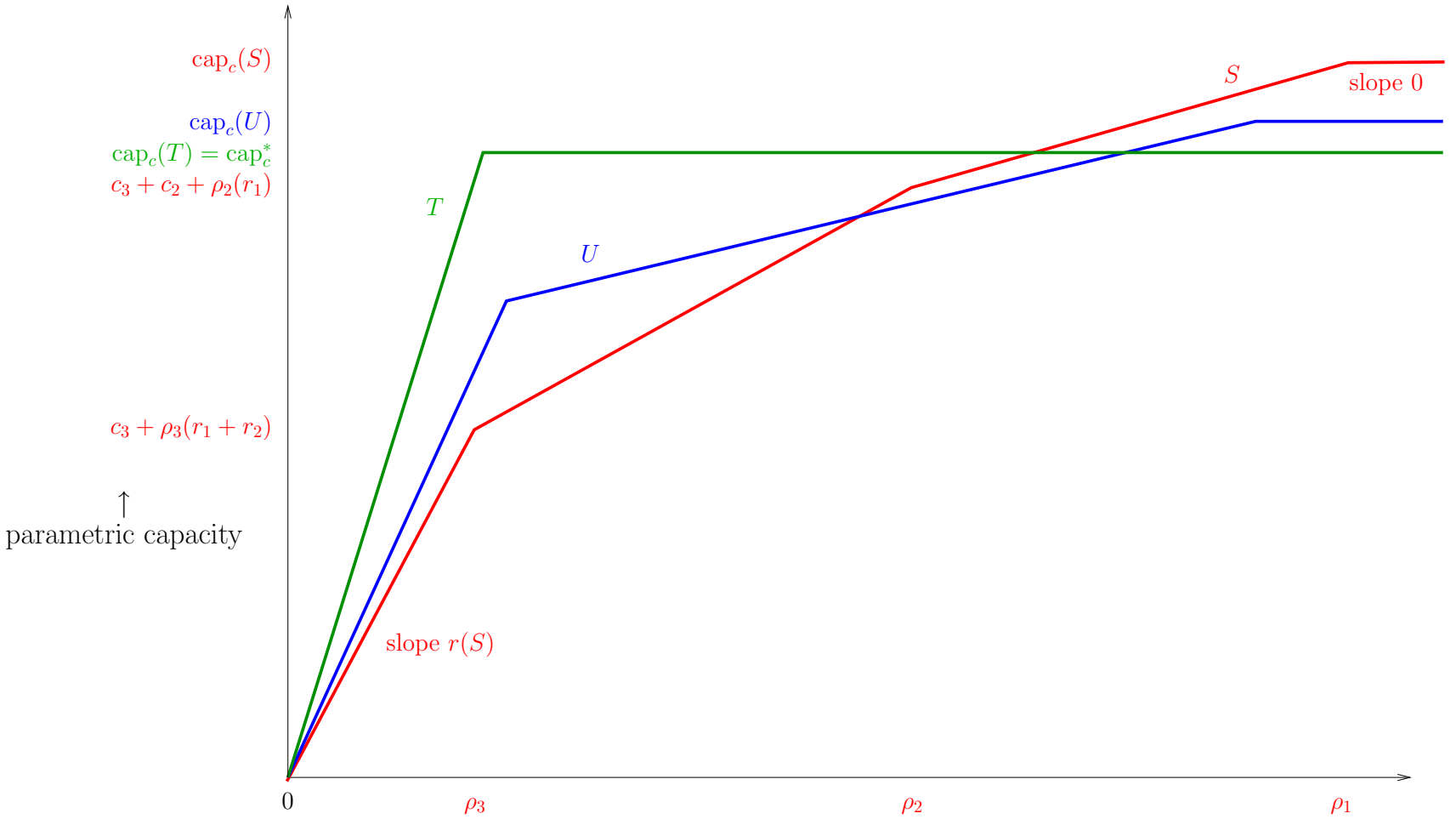
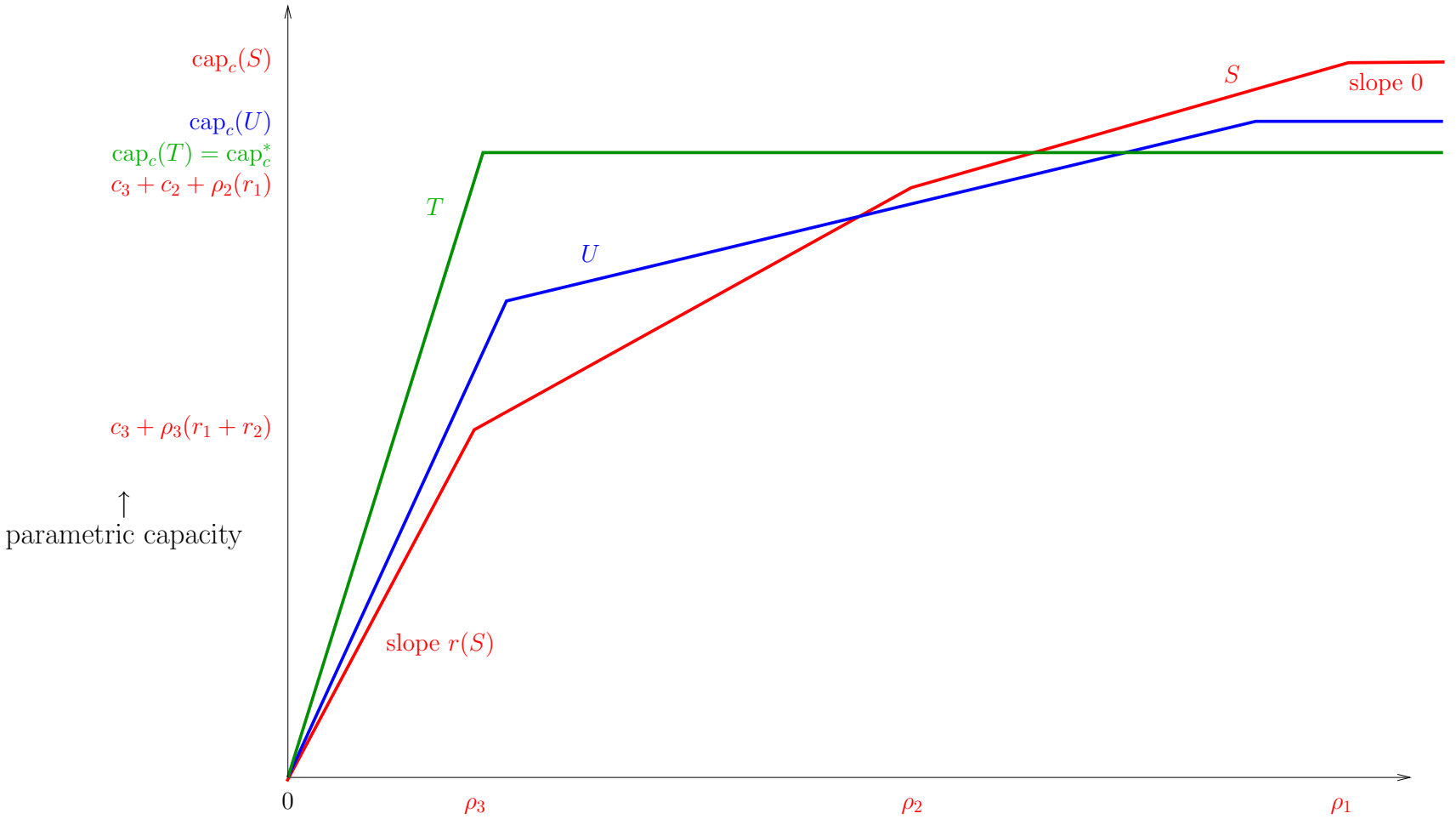


The overall parametric capacity curve: the λ-profile

For a fixed value of λ, we want to find the S whose parametric capacity at λ is minimum, so we just want the pointwise minimum of all these curves.



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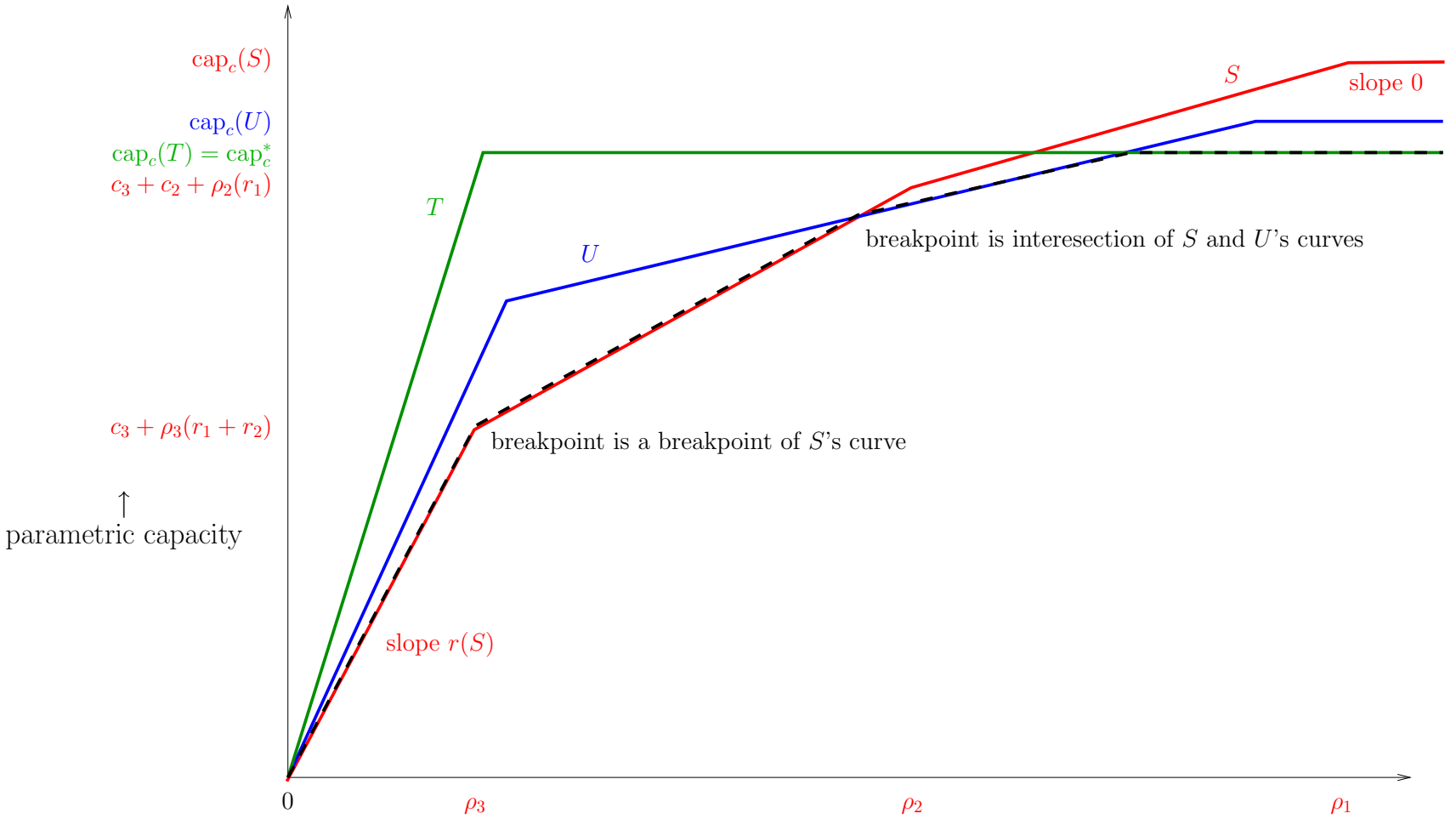
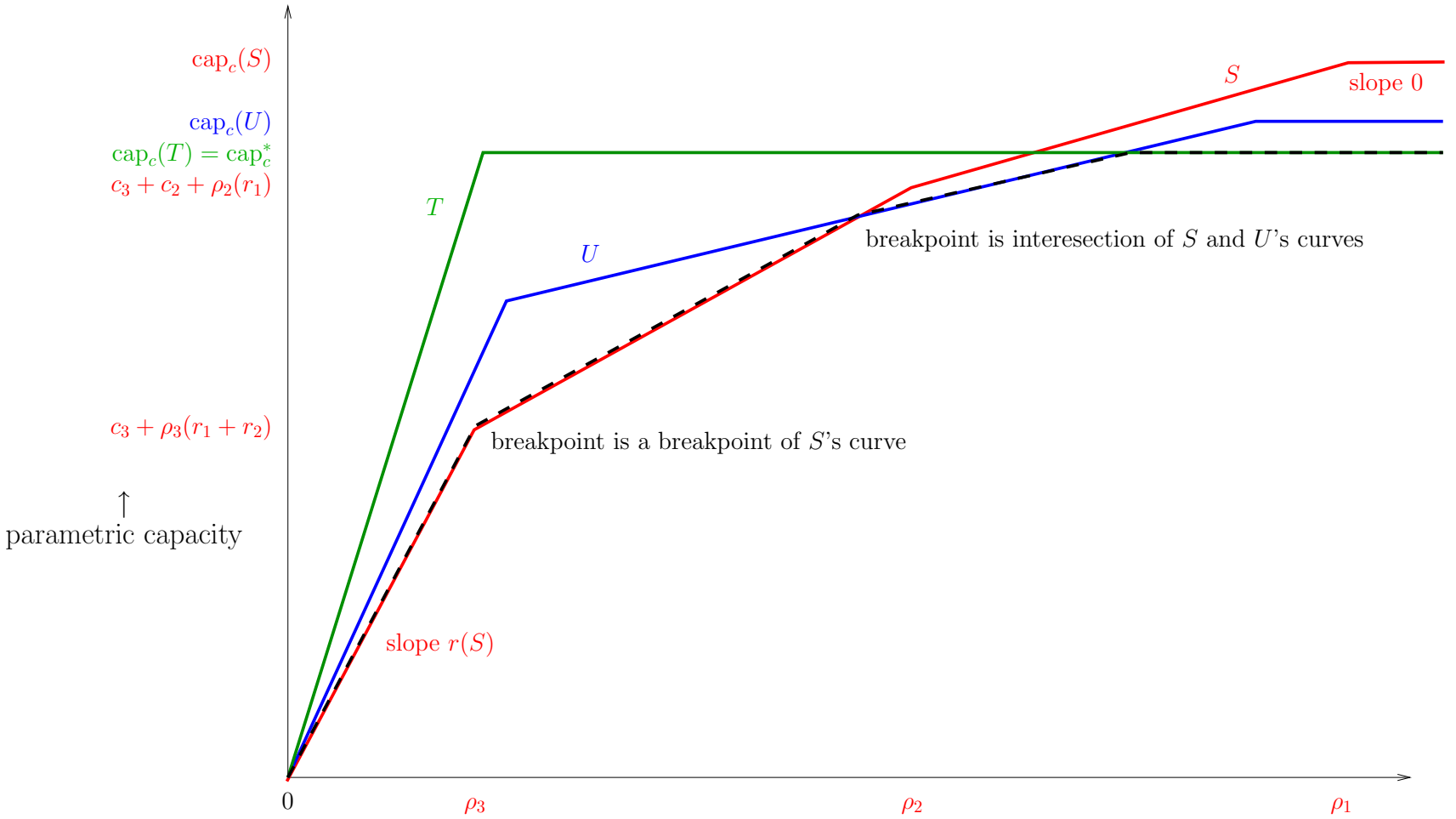


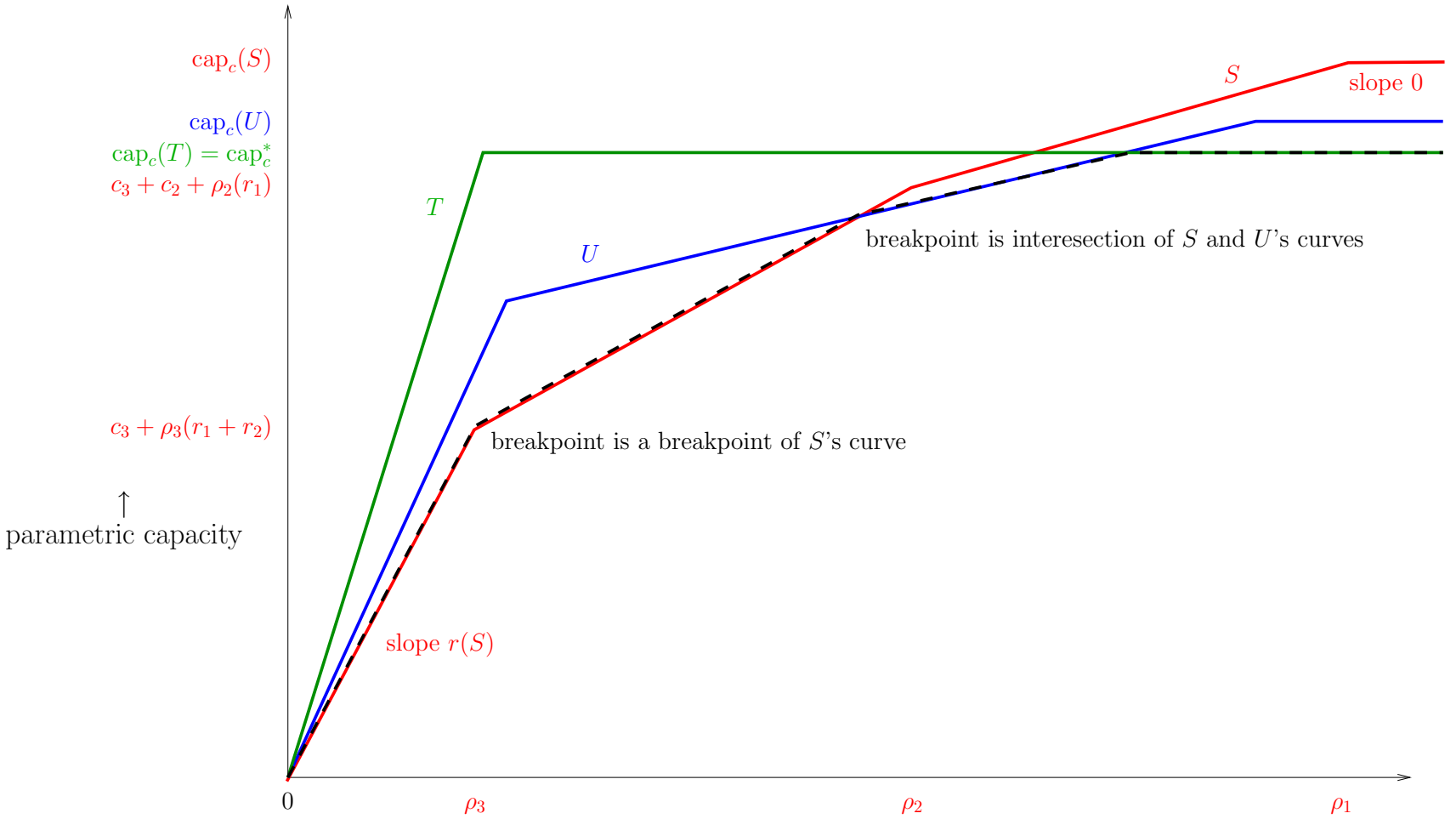
The overall parametric capacity curve: the λ-profile

Since the minimum of a bunch of concave curves is again concave, this time we do not need to linearize. We call this overall parametric capacity curve the λ-profile.



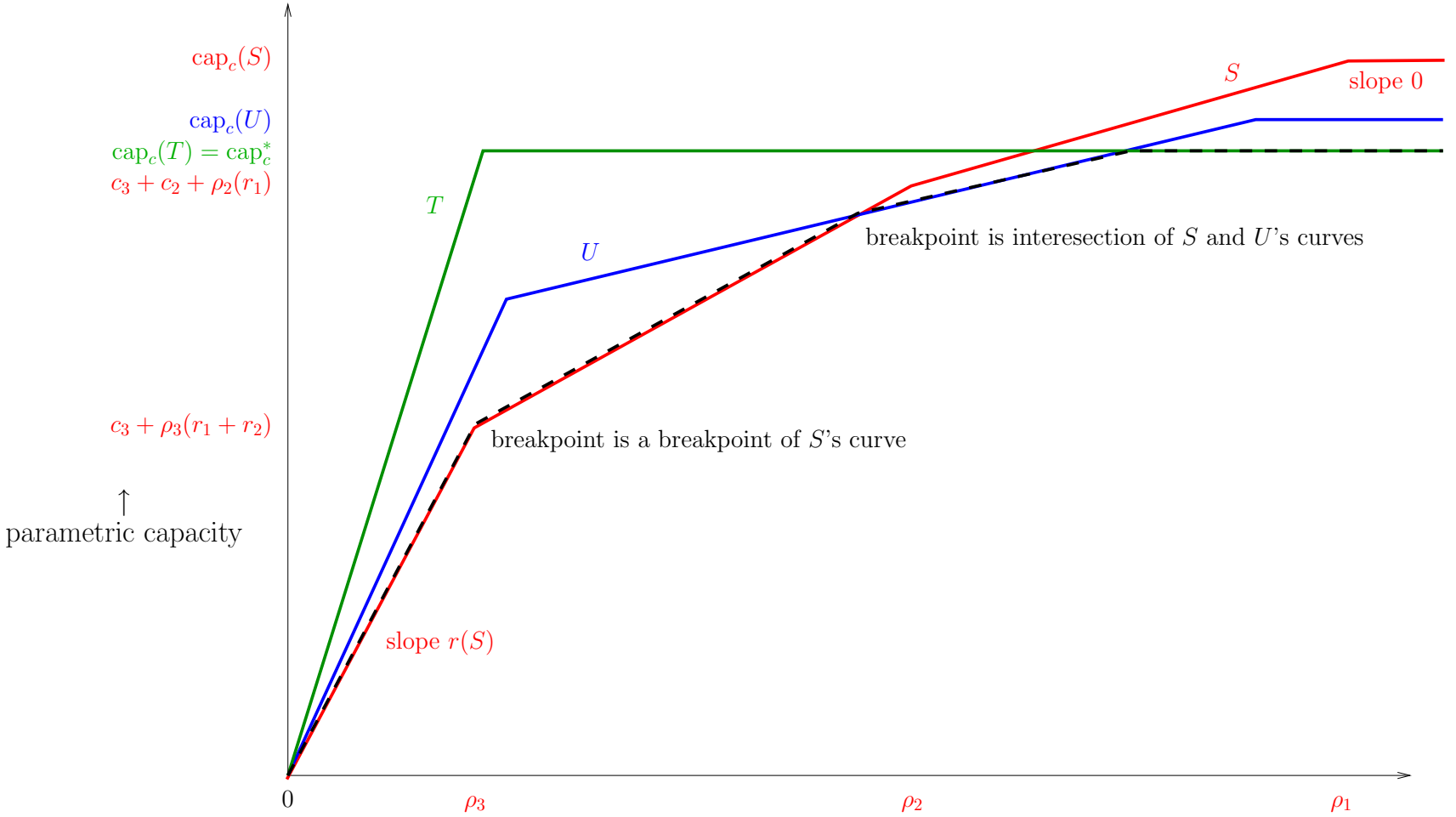
The overall parametric capacity curve: the λ-profile

We can compute things like $\mathrm{cap}^*(\lambda)$ easily using parametric min cut technology.



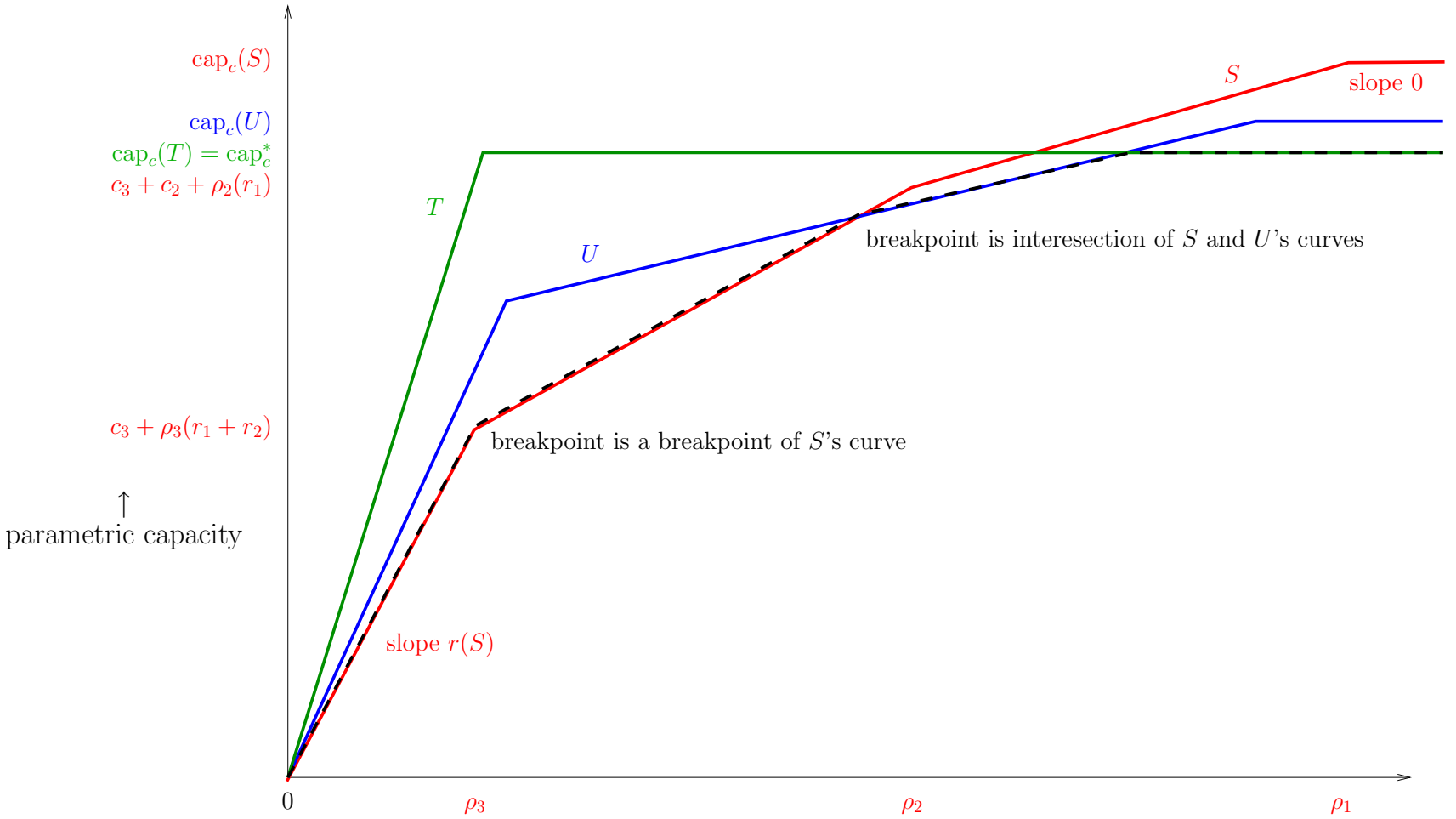
The overall parametric capacity curve: the λ-profile

We can show that the conjugate duality between S's interdiction and parametric capacity curves carries over to conjugate duality between the B-profile and the λ-profile.



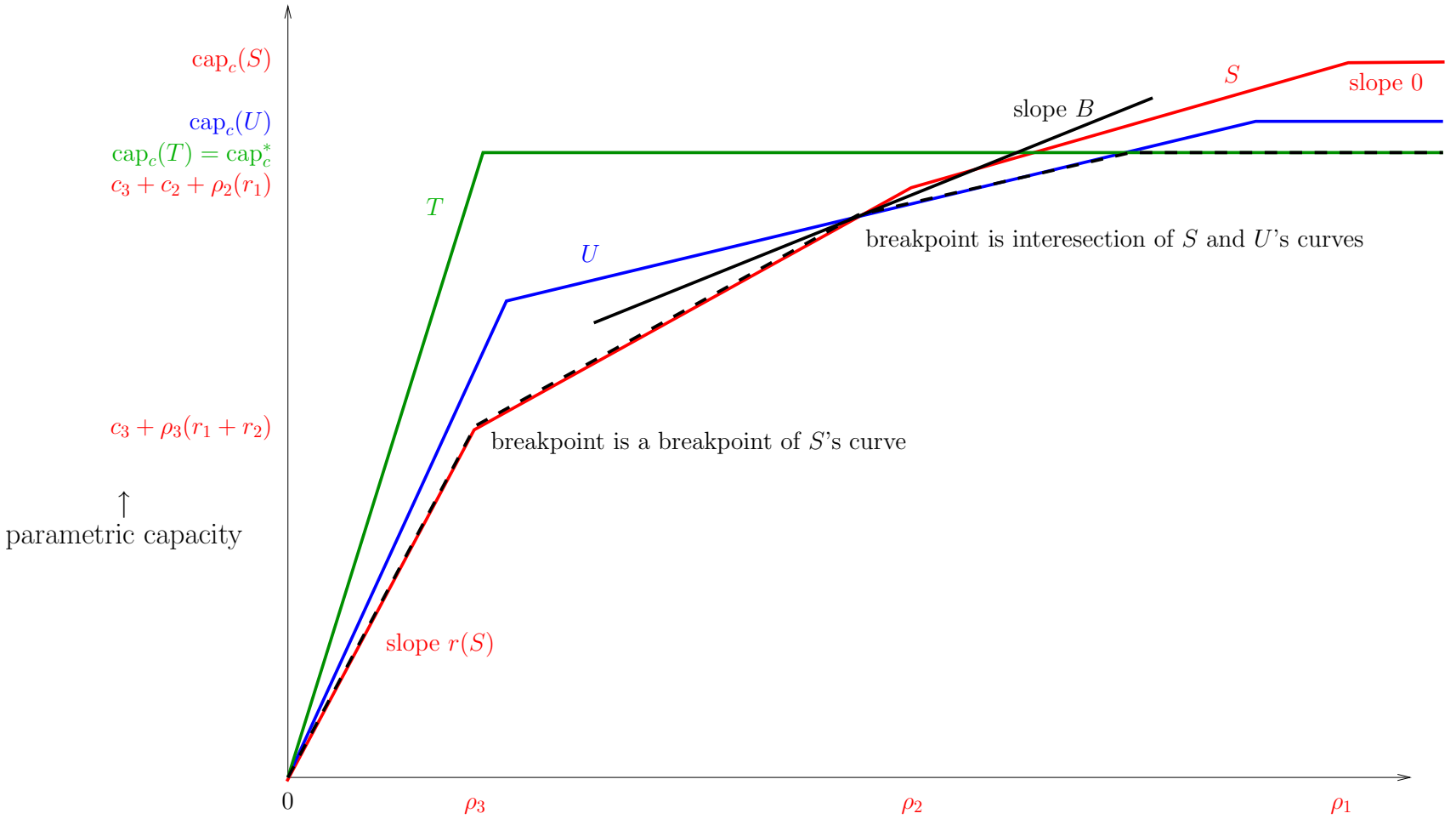
The overall parametric capacity curve: the λ-profile

Recall that to get our pseudo-approximation for a given B, we want to compute the two cuts S_1 and S_2 bracketing B on the B-profile.



The overall parametric capacity curve: the λ-profile

Conjugate duality implies that this is equivalent to finding a breakpoint on the λ-profile whose adjacent slopes bracket B, here S and U.



Outline

The key breakpoint subproblem

- Notice that any breakpoint $\hat{\lambda}$ of the λ-profile is defined by the intersection of a segment to its left coming from cut $S^-(\hat{\lambda})$ with local slope $\mathrm{sl}^-(\hat{\lambda})$, and a segment to its right coming from cut $S^+(\hat{\lambda})$ with local slope $\mathrm{sl}^+(\hat{\lambda})$, with $\mathrm{sl}^-(\hat{\lambda}) > \mathrm{sl}^+(\hat{\lambda})$ by concavity.

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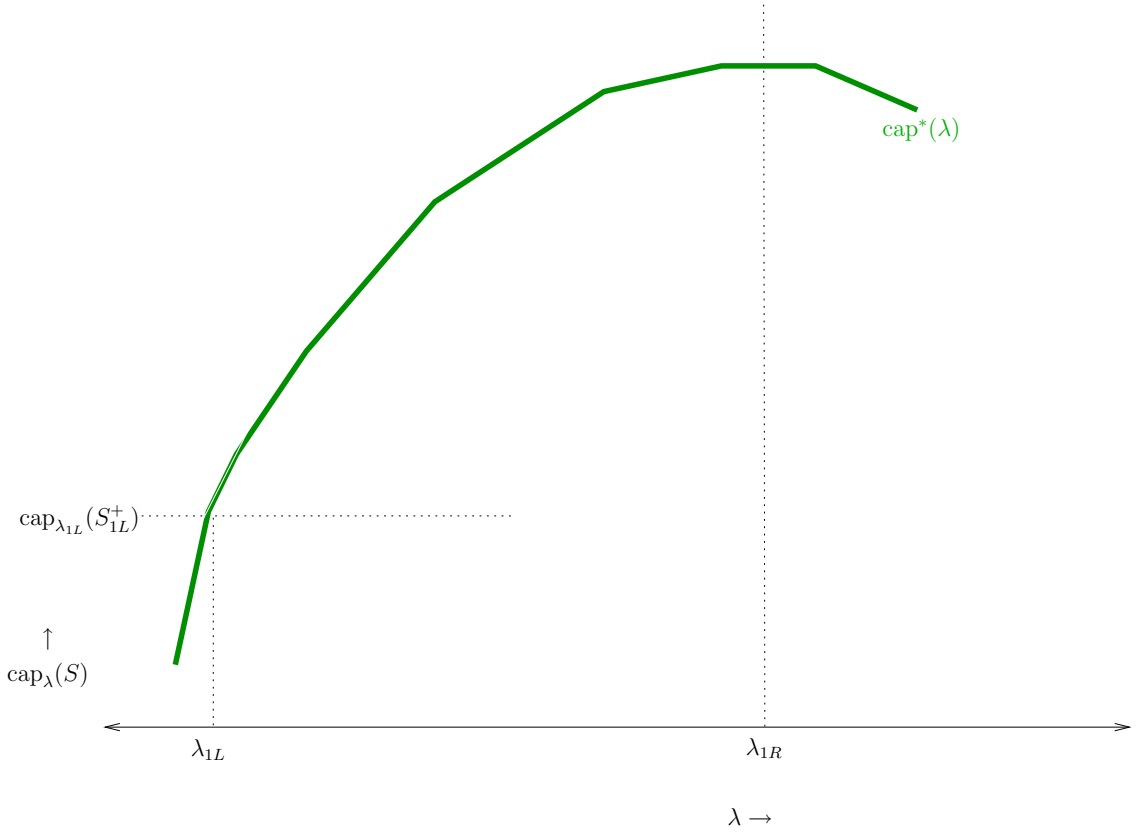
- Can we do better?

Discrete Newton gives a better algorithm

Set $\lambda_L = 0$ and $\lambda_R = \text{cap}_r^*$ as before. Denote $\text{sl}^+(\lambda_L)$ by sl_L^+ and $\text{sl}^-(\lambda_R)$ by sl_R^-.



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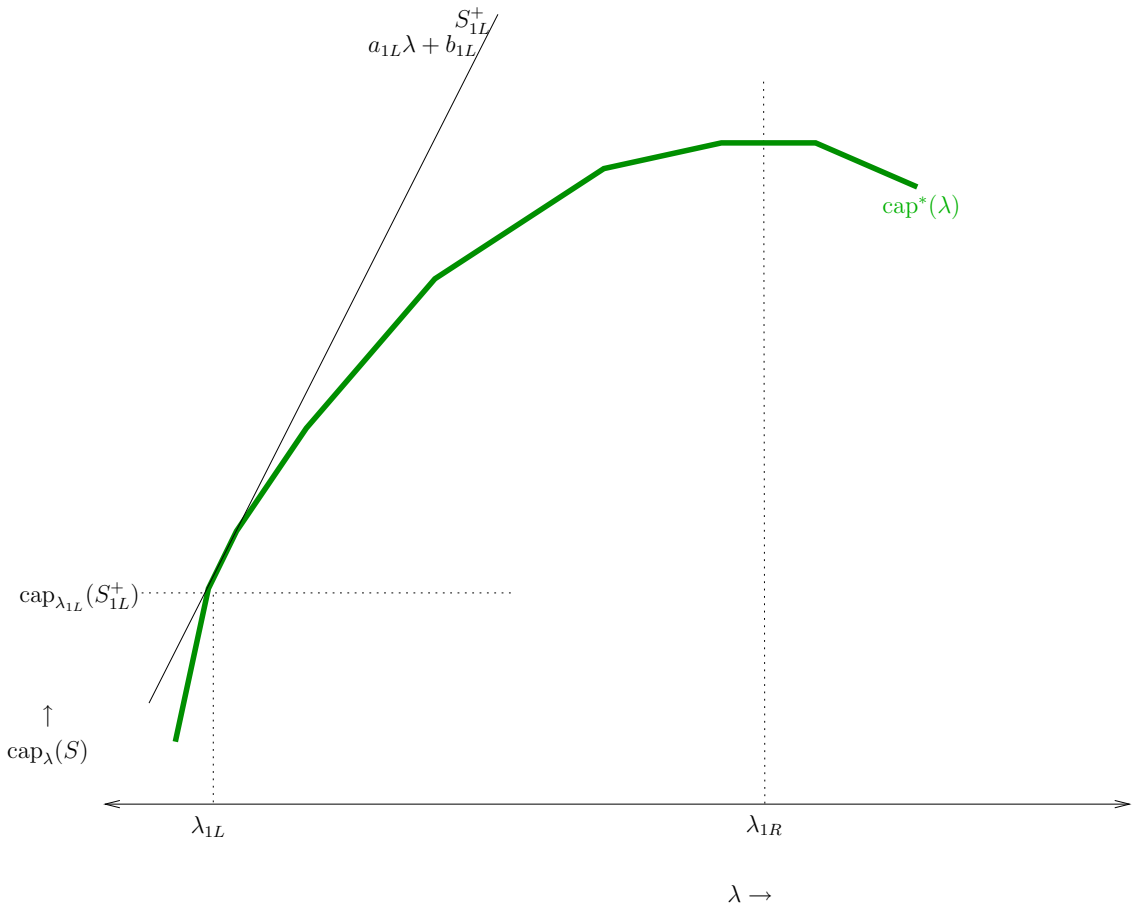

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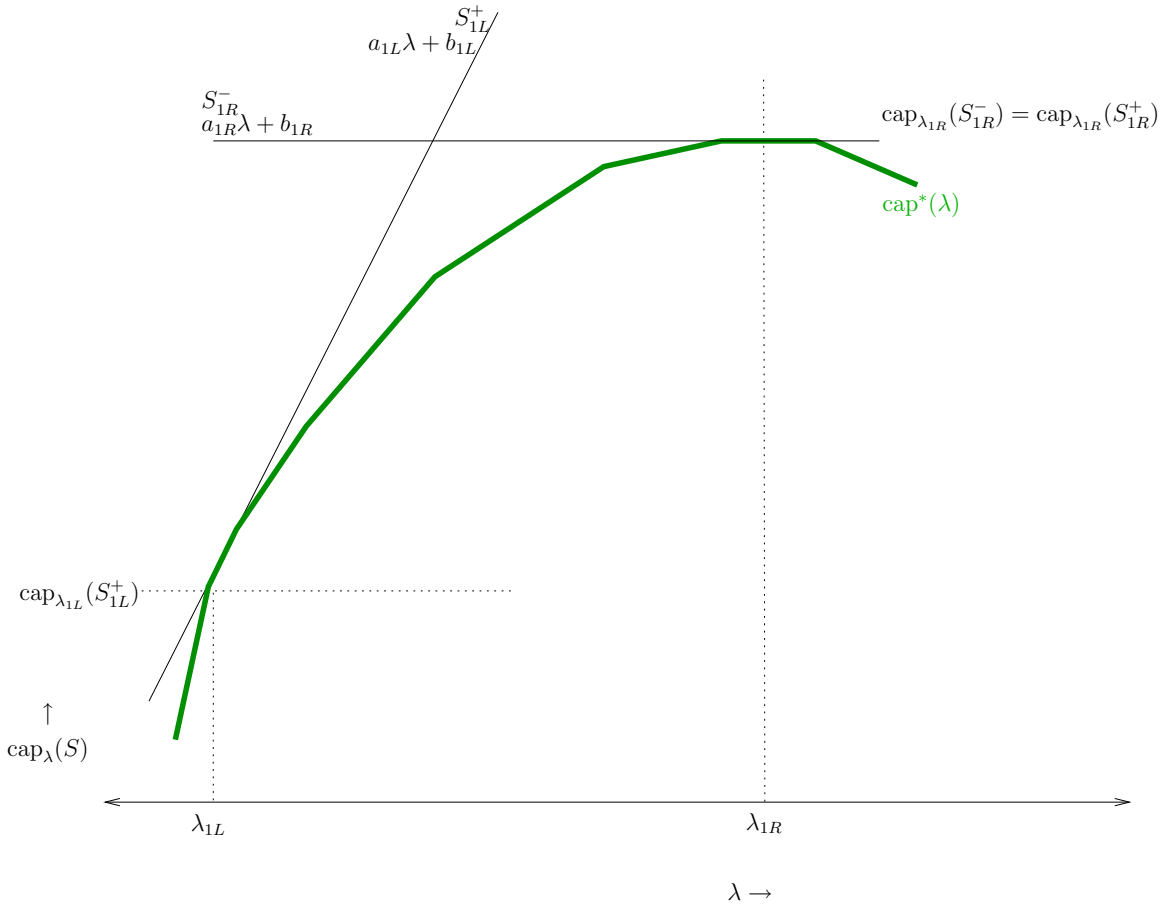

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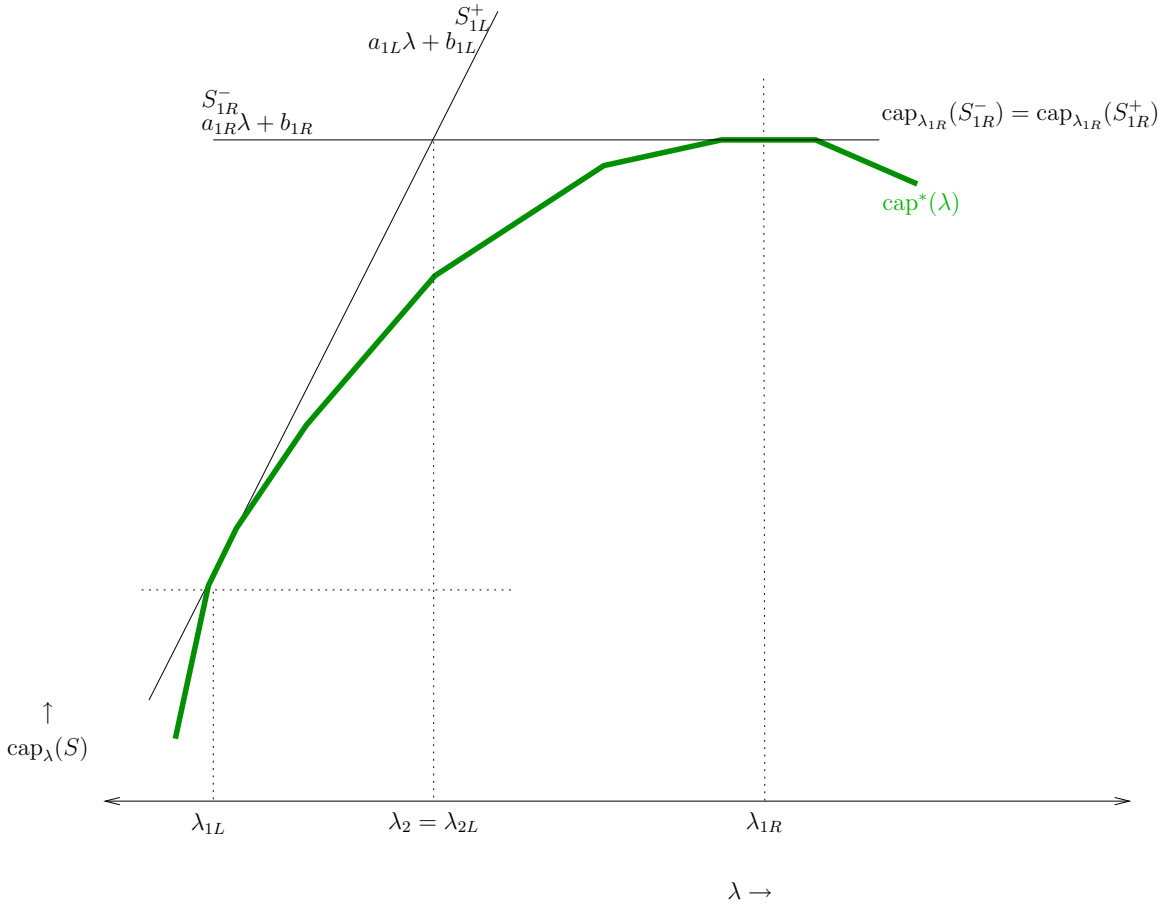
Compute $\hat{\lambda}$ as the intersection of the line of slope sl_L^+ through $(\lambda_L, \mathrm{cap}^*(\lambda_L))$, and the line of slope sl_R^- through $(\lambda_R, \mathrm{cap}^*(\lambda_R))$, a max flow w.r.t. $\hat{\lambda}$, and $\mathrm{sl}^-(\hat{\lambda})$ and $\mathrm{sl}^+(\hat{\lambda})$.



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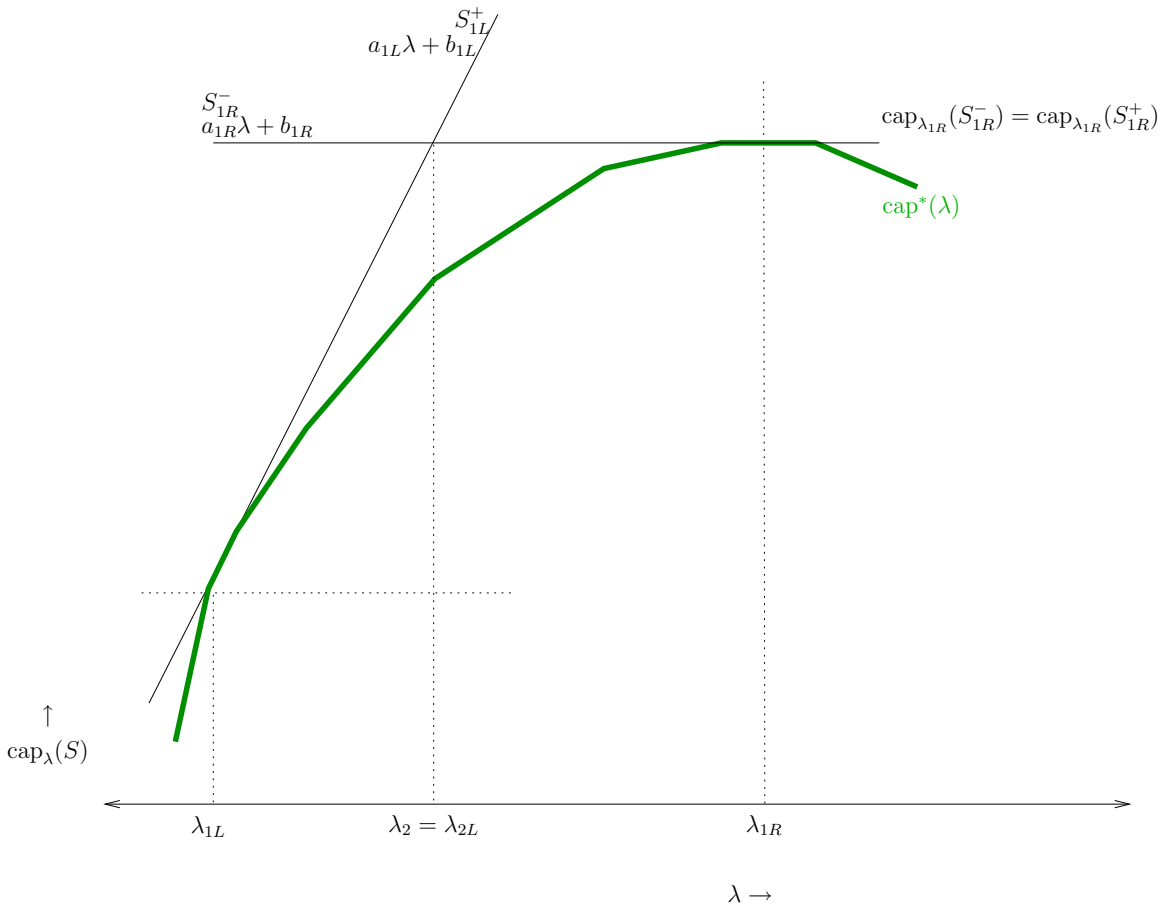

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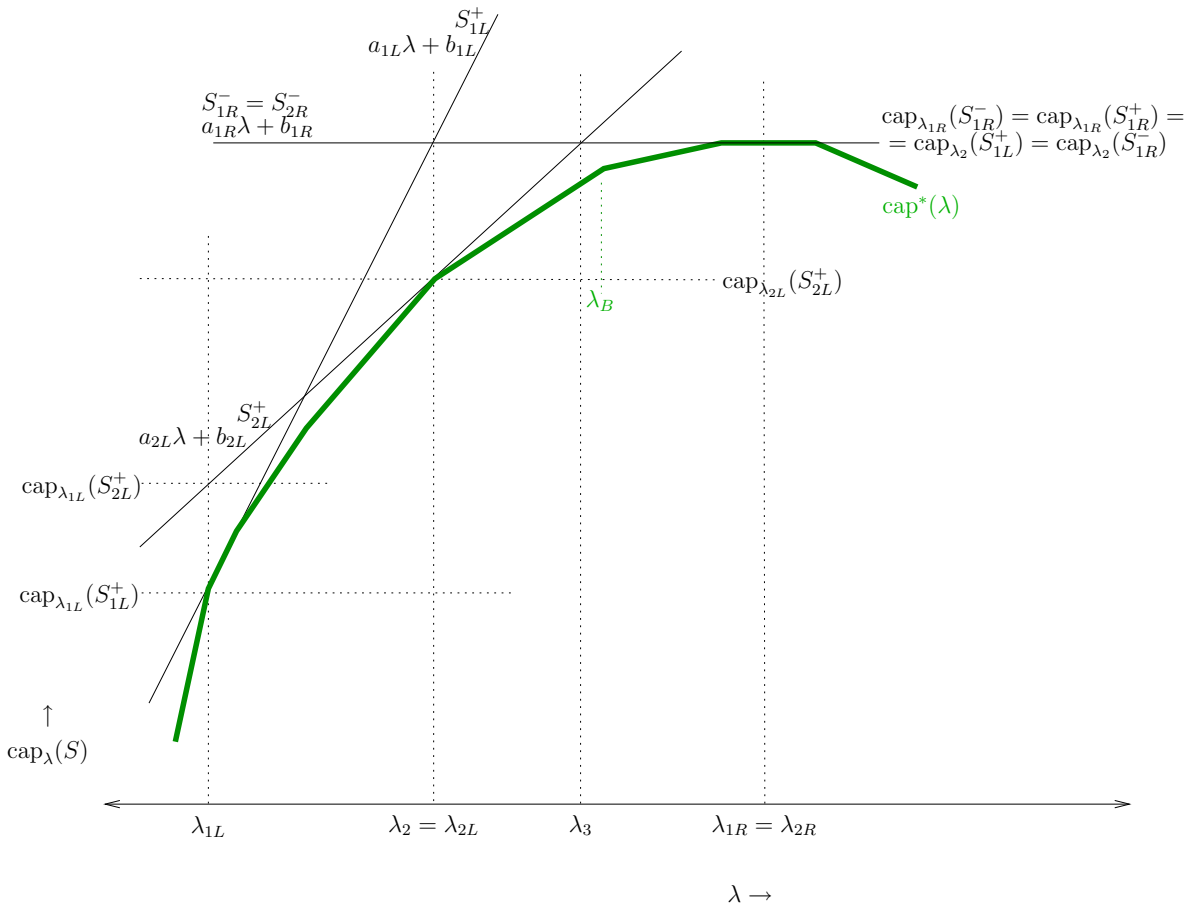

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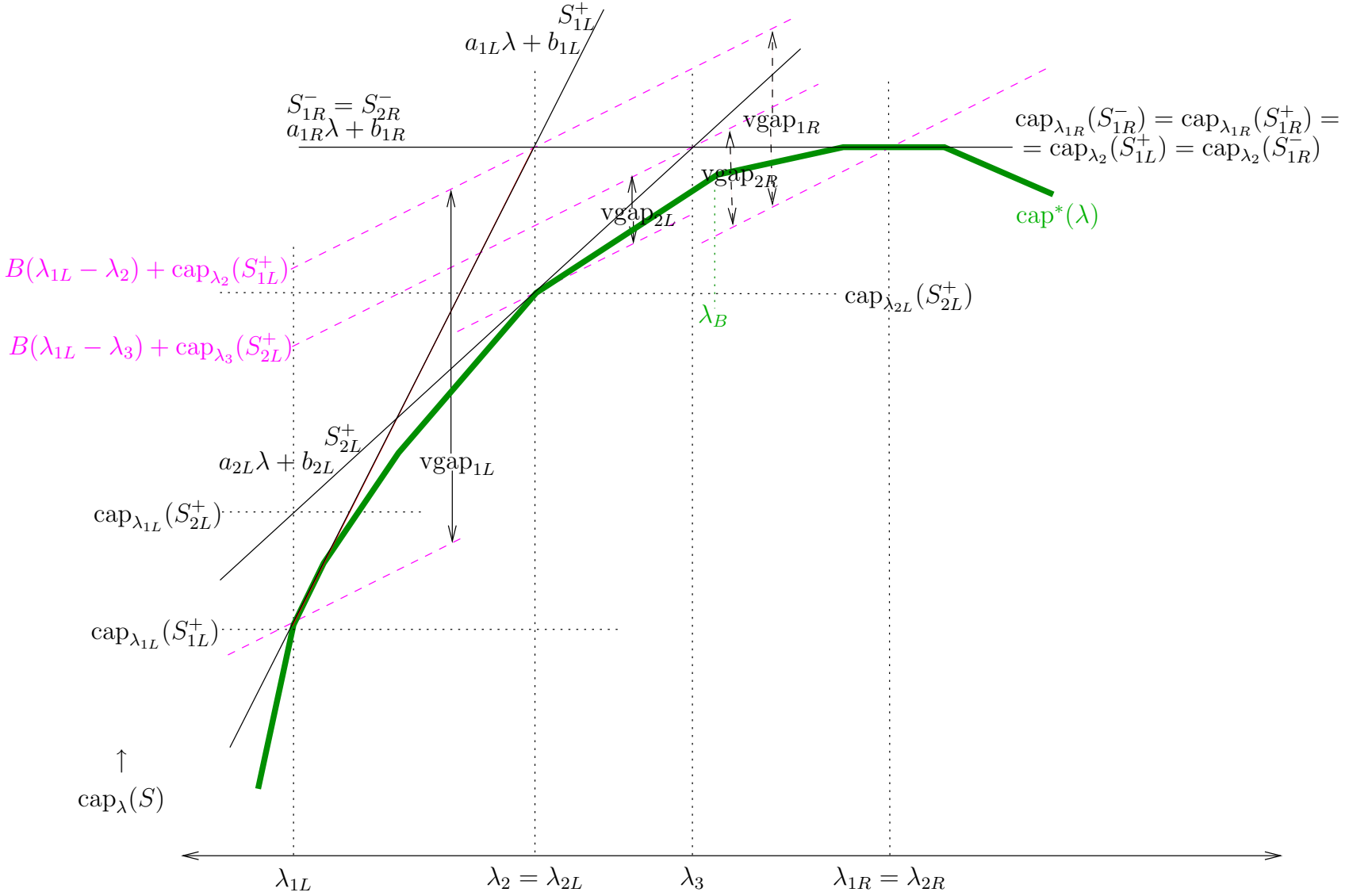
- Also define slgap_L to be $\mathrm{sl}_L^+ - B$ and slgap_R to be $B - \mathrm{sl}_R^-$.

vgap illustrated



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- (1) was originally proved in Mc+Ervolina '94. Then Rote, and Radzik '92–'98 showed that Newton-B is sometimes faster than Binary Search, and has a strongly polynomial bound.

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- Indeed, this Newton-B algorithm and its analysis works for any concave (or convex) function, even continuous ones.

Outline

Multiple budgets equals multiple parameters

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- Now we'd be trying to find a point on the parametric surface whose local derivatives bracket the given budgets in the coordinate directions.
- As before we could solve this via LP, but we'd prefer a combinatorial algorithm.
- Interdiction already gets complicated with two parameters, so let's consider a simpler multiple parameter scheduling problem instead.

Chen's '94 scheduling problem

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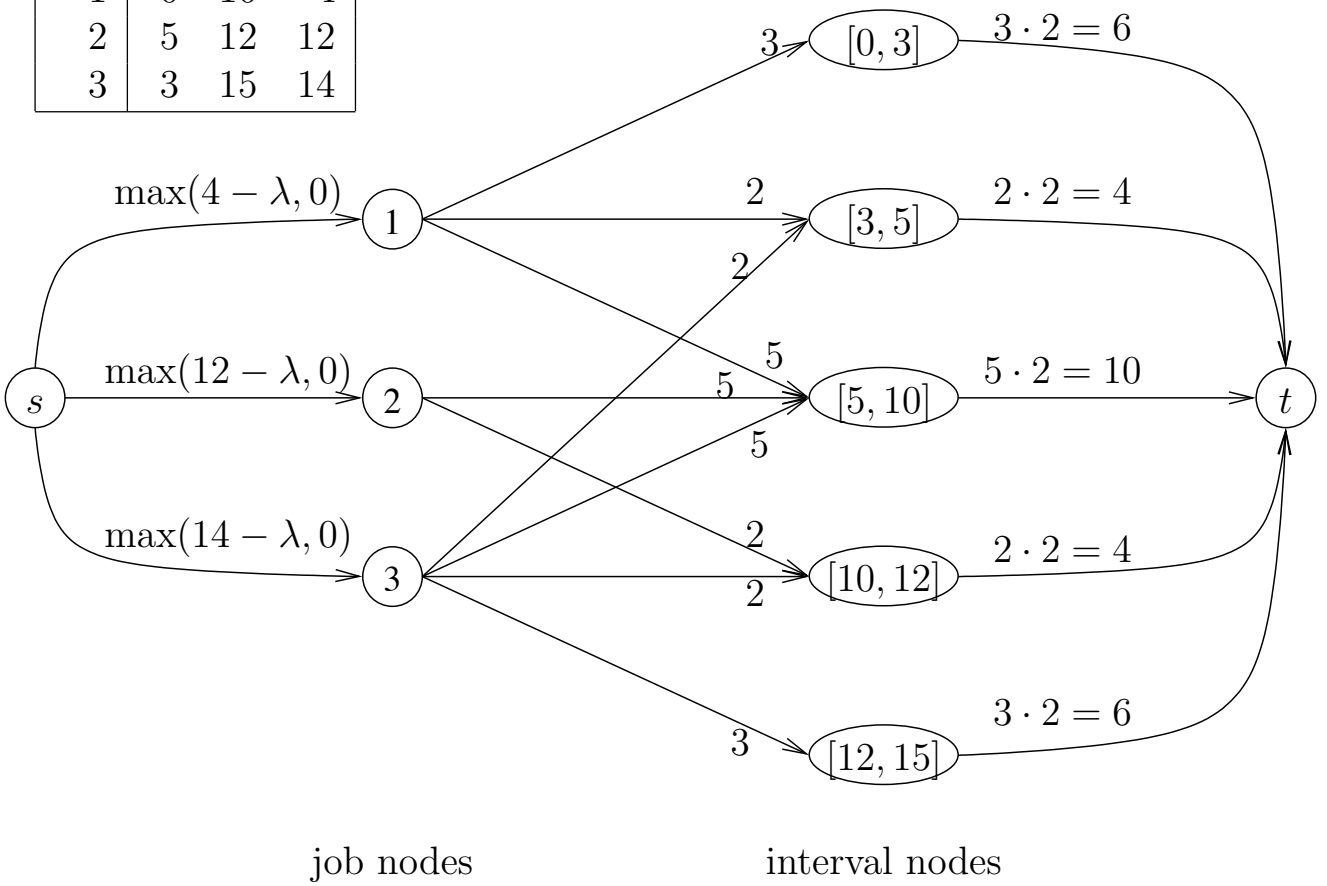
- Now we want to minimize λ such that there exists a flow saturating all residual job arcs.

Chen's scheduling problem: example

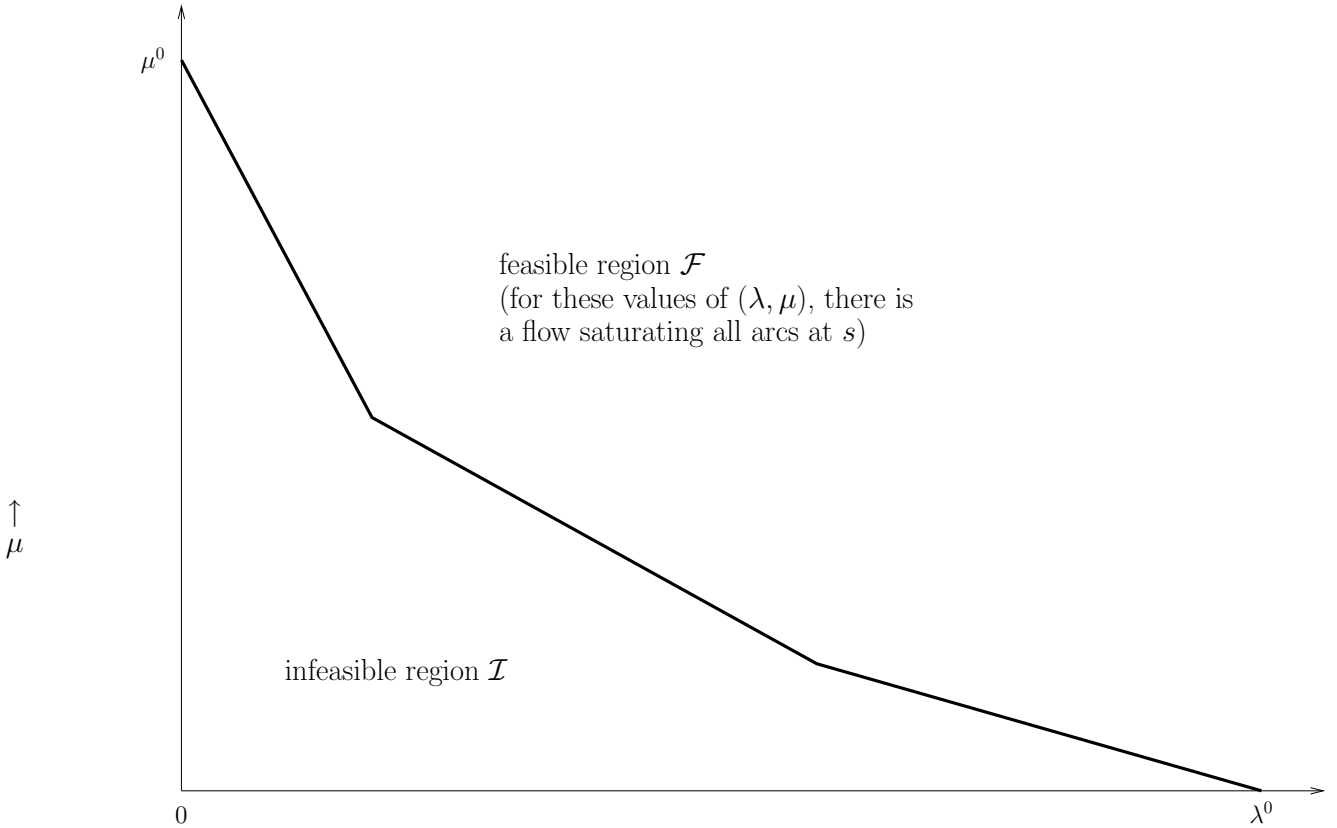
This single-parameter version can be solved using Gallo-Grigoriadis-Tarjan (GGT) '89 parametric min cut in $O(1)$ max flow time.

job	r_j	d_j	p_j
1	0	10	4
2	5	12	12
3	3	15	14



Two-parameter Chen

Suppose now that there are two ways to outsource, λ and μ such that if we pay $\$\lambda + \$\mu$, we reduce p_j to $\max(0, p_j - a_j\lambda - b_j\mu)$.
In the (λ, μ) plane there is a piecewise linear convex curve separating feasible points from infeasible ones.



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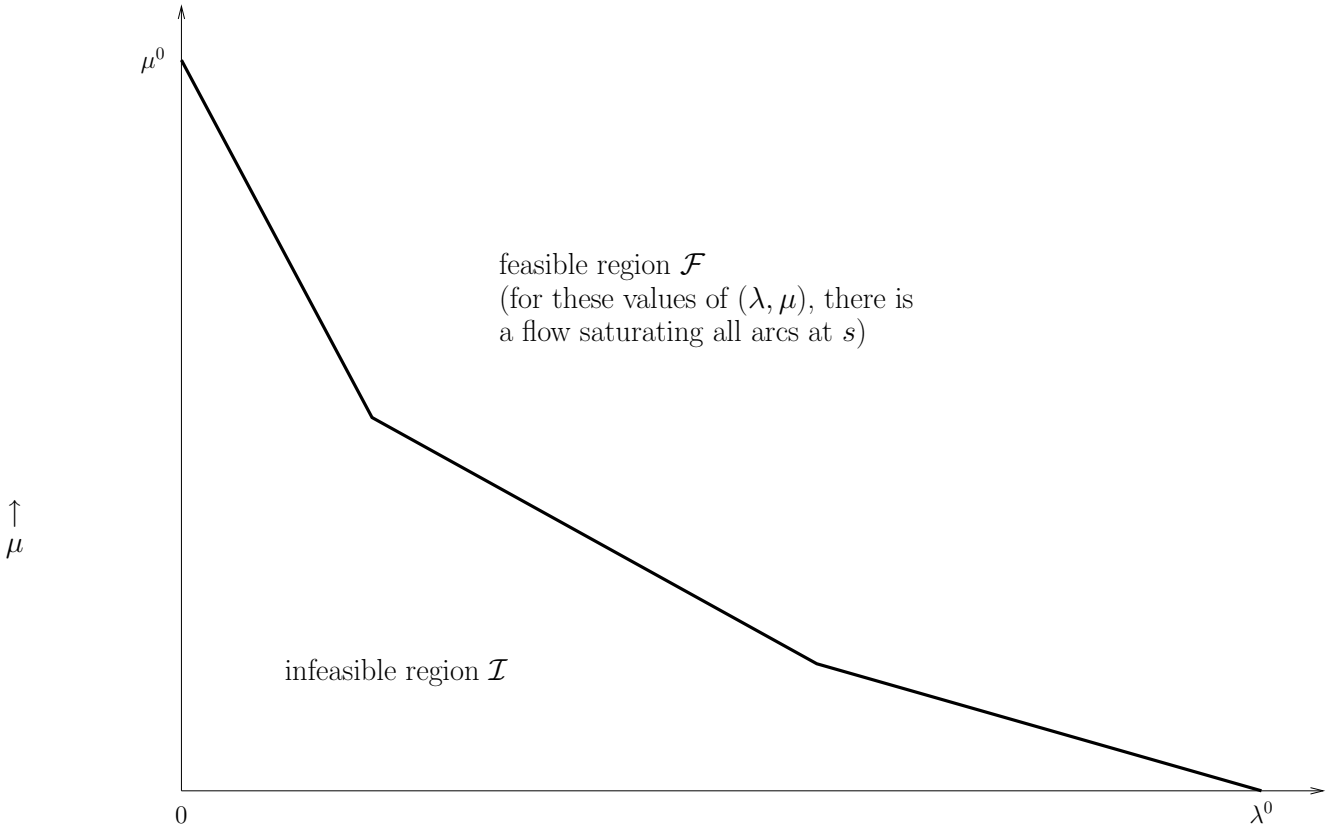

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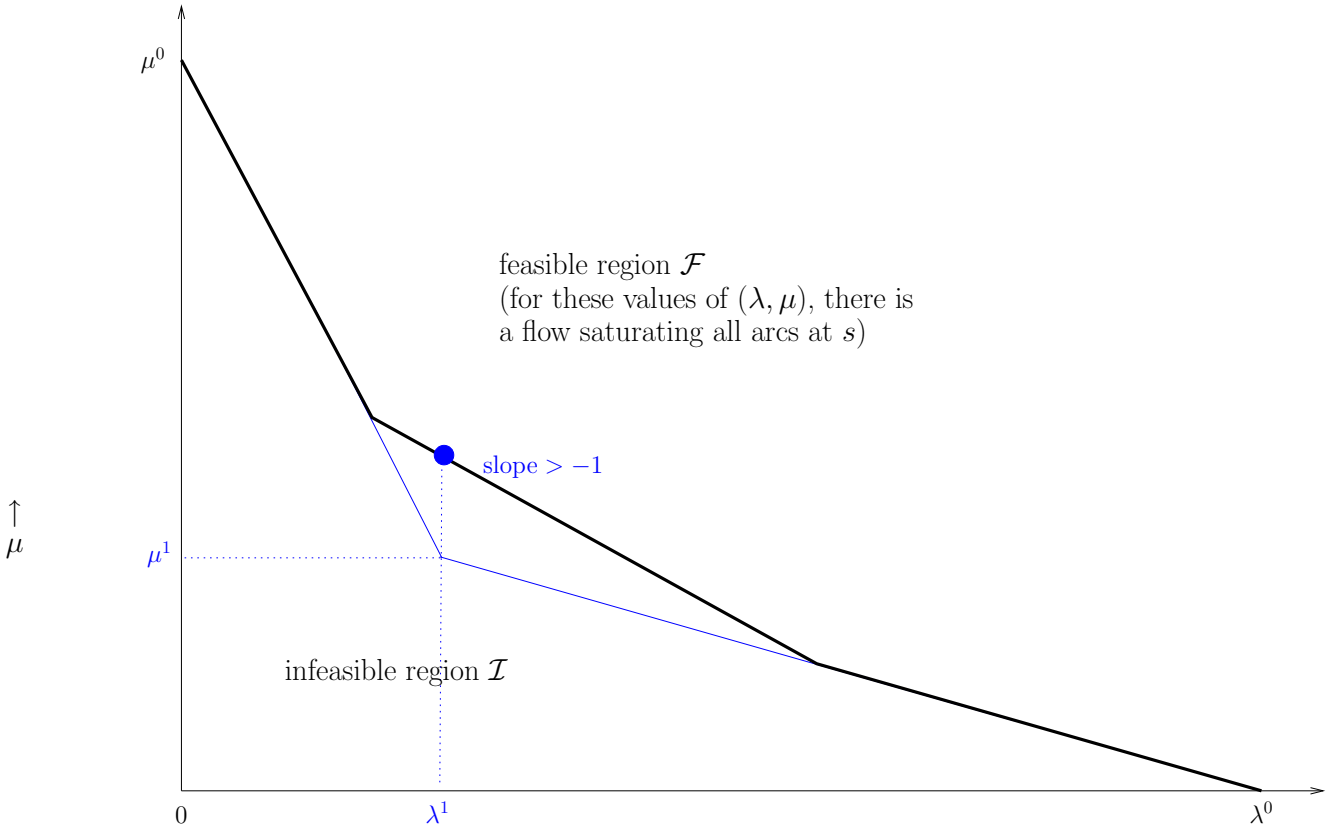

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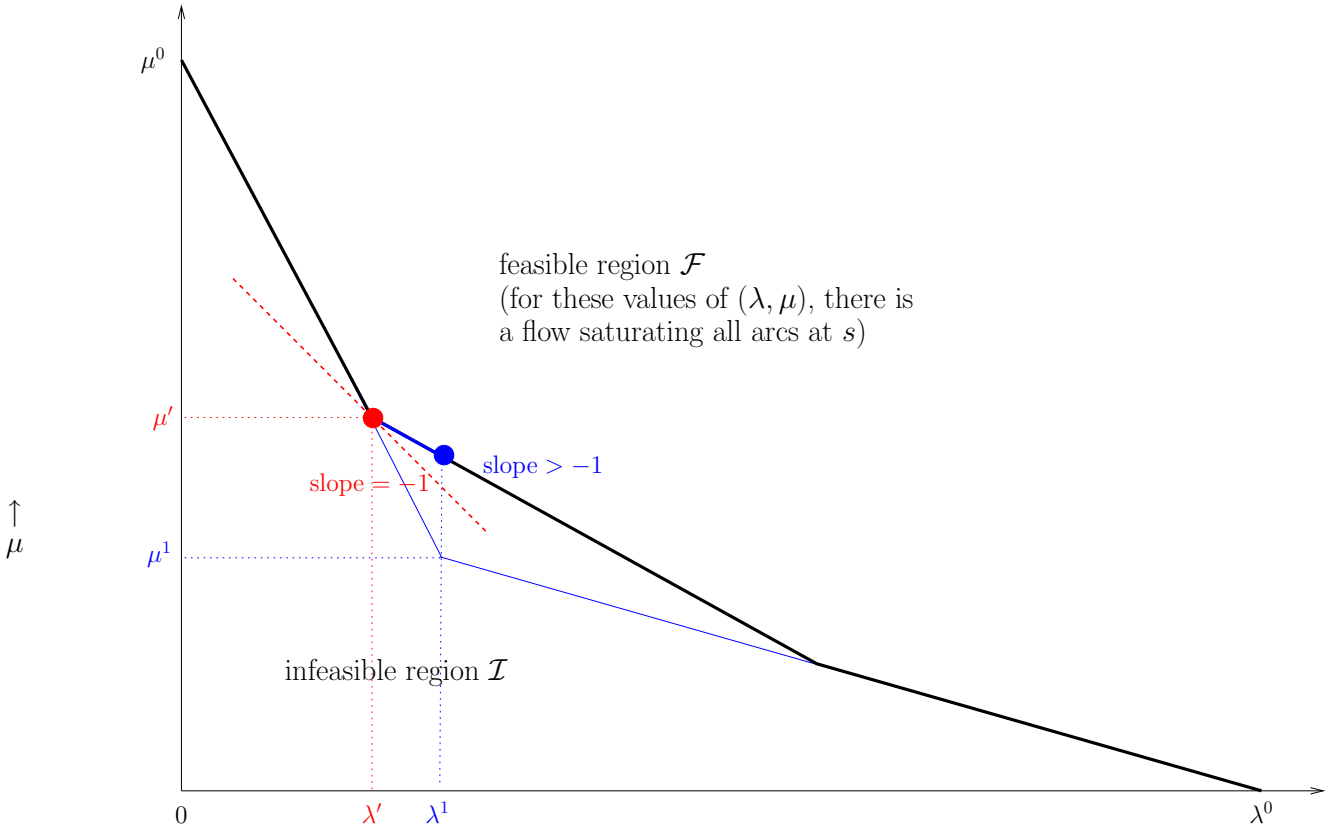

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- This generalizes to any fixed number of parameters.
- Open Question: LP is polynomial even when the number of parameters is not fixed. Can we get a combinatorial algorithm then?

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- Open Question: When capacities are (piecewise) linear, how many different min cuts can we have over all (λ, μ)?

Any questions?

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Comments?