Perfect Packings in Quasirandom Hypergraphs

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Joint work with Dhruv Mubayi

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Quasirandom Graphs

Fix $0 < p < 1$. Let $\mathcal{G} = \{G_n\}_{n \to \infty}$ be a sequence of graphs with $|V(G_n)| = n$ and $|E(G_n)| = p\binom{n}{2} + o(n^2)$.

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Theorem (Thomason 1987, Chung-Graham-Wilson 1989)

$\texttt{Disc}_p$ and $\texttt{Count}_p\texttt{[All]}$ are equivalent.

Quasirandom Hypergraphs

Observation (Rödl)

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Proof.

Use Erdős and Hajnal's construction: let T be a random graph tournament and form a three-uniform hypergraph by making each cyclically oriented triangle a hyperedge. There is no $K_4^{(3)}$ but $\mathtt{Disc}_{1/4}$ holds. □

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Theorem (Komlós-Sárközy-Szemerédi 2001 – Alon-Yuster Conjecture)

For every F and G where $|V(F)|$ divides $n = |V(G)|$ and $\delta(G) \geq (1 - 1/\chi(F))n + C_F$, G contains a perfect F-packing.

Perfect Packings in Hypergraphs

Theorem (Rödl-Ruciński-Szemerédi 2009, Kühn-Osthus 2006)

If H is a k-uniform hypergraph, k divides $n = |V(H)|$, and $\delta_{codeg}(H) \geq n/2 - k + C$, then H has a perfect matching where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k.

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Other results for various hypergraphs F are known, including K_4 (Keevash-Mycroft, Lo-Markström, Pikhurko), K_4^- (Lo-Markström), $K_4 - 2e$ (Kühn-Osthus, Czygrinow-DeBiasio-Nagle)

Quasirandomness and Perfect Packings

Let $\mathcal{G} = \{G_n\}_{n \to \infty}$ be a sequence of graphs. We say that $\mathcal{G}$ has a perfect F-packing if all but finitely many of the graphs G_n with $|V(F)|$ dividing n have a perfect F-packing.

Theorem (Komlós-Sárközy-Szemerédi 1997)

Let $0 < p < 1$ be fixed and let F be any graph. Let $\mathcal{G}$ be a graph sequence satisfying $\mathtt{Disc}_p$ with $\delta(G_n) = \Omega(n)$. Then $\mathcal{G}$ has a perfect F-packing.

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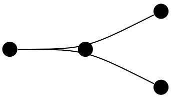
Problem

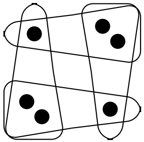
Characterize the 3-uniform hypergraphs F for which for all $0 < p < 1$ a hypergraph sequence $\mathcal{H}$ satisfying $\mathtt{Disc}_p$ with $\delta(H_n) = \Omega(n^2)$ is forced to have a perfect F-packing.

Some Hypergraphs

- A hypergraph F is *linear* if every pair of distinct edges share at most one vertex.

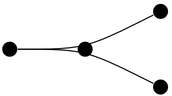


Cherry



$(2,1)$-four-cycle

Our Results - Perfect Packings



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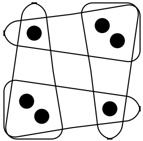

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Theorem (L-Mubayi)

Fix $0 < p < 1$. Let $\mathcal{H}$ be a 3-uniform hypergraph sequence satisfying $\mathtt{Disc}_p$. Then $\mathcal{H}$ has a perfect F-packing if

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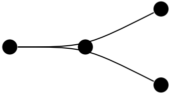

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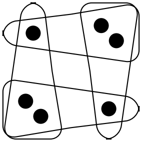

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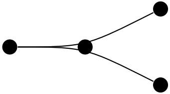

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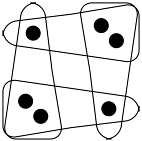

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Theorem (L-Mubayi)

Fix $0 < p < 1$. Let $\mathcal{H}$ be a 3-uniform hypergraph sequence satisfying $\mathtt{Disc}_p$. Then $\mathcal{H}$ has a perfect F-packing if

- F is linear and $\delta(H_n) = \Omega(n^2)$,
- F is the cherry and $\delta_{codeg}(H_n) = \Omega(n)$,
- F is the $(2, 1)$-four-cycle and $\delta_{codeg}(H_n) = \Omega(n)$.

Our Results - Constructions

The Erdős-Hajnal construction satisfies $\texttt{Disc}_{1/4}$, has minimum codegree $(1 - o(1))\frac{n}{4}$, and has no perfect K_4^--packing.

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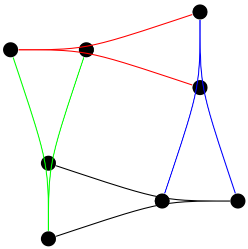
Theorem (L-Mubayi)

There exists a 3-uniform hypergraph sequence $\mathcal{H}$ satisfying $\texttt{Disc}_{1/8}$ with $\delta_{codeg}(H_n) \geq (1 - o(1))\frac{n}{8}$ and has no perfect cherry-four-cycle packing. The cherry four cycle is the following hypergraph:



Sparse Setting

Theorem (Krivelevich-Sudakov 2002)

If G is a regular, n-vertex graph with

$$\lambda_2(G) \leq \frac{(\log \log n)^2}{1000 \log n \, (\log \log \log n)} \lambda_1(G)$$

and n is large, then G is Hamiltonian.

Eigenvalue definitions of Friedman and Wigderson

- Let H be a 3-uniform, n-vertex hypergraph. The *adjacency map of H* is

$$\tau : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$\tau(e_x, e_y, e_z) = \begin{cases} 1 & \text{if } xyz \in E(H) \\ 0 & \text{otherwise} \end{cases}$$

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- Let $J : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the all-ones map.

$$\lambda_2(H) := \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|=1}} \left| \tau(x,x,x) - \frac{k!|E(H)|}{n^k} J(x,x,x) \right|$$

Theorem (L-Mubayi)

Let H be a 3-uniform, n-vertex hypergraph with n large and let $p = |E(H)|/\binom{n}{3}$. If n is divisible by three, $\delta_{codeg}(H) \geq \frac{pn}{100}$, and

$$\lambda_2(H) \leq Cp^{15/2}\lambda_1(H),$$

then H contains a perfect matching.

Absorbing Sets

To prove the existence of perfect F-packings, we use the absorption method of Rödl-Ruciński-Szemerédi.

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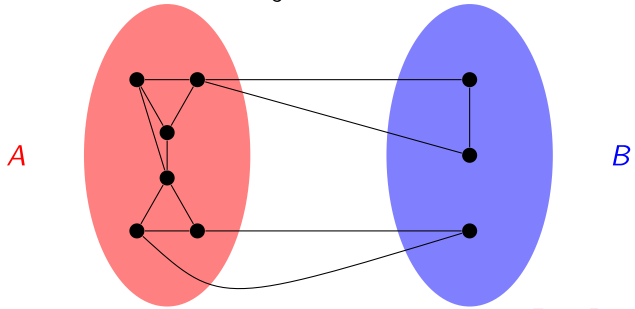
A K_3-absorbs B



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- For K_4 and the cherry four-cycle, what values of p cause $\texttt{Disc}_p$ and linear min codegree to imply a perfect packing?
- What about other hypergraph quasirandom properties besides $\texttt{Disc}_p$? What can they pack?