Automorphism breaking in locally finite graphs

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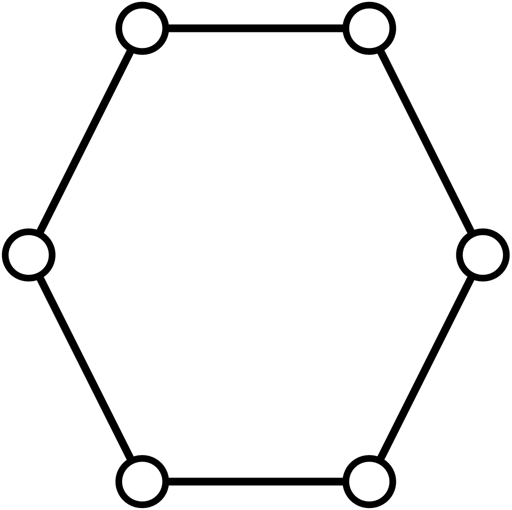
Distinguishing Graphs



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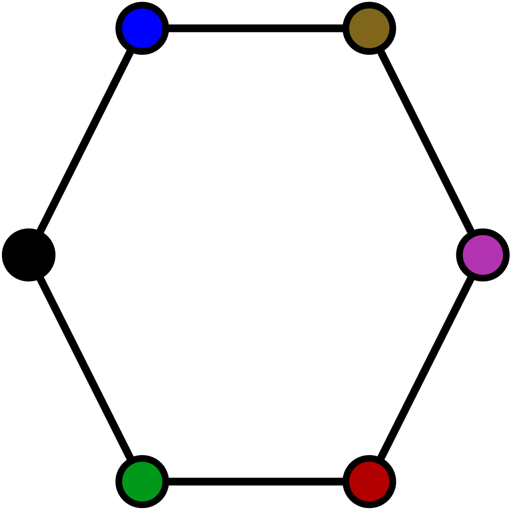

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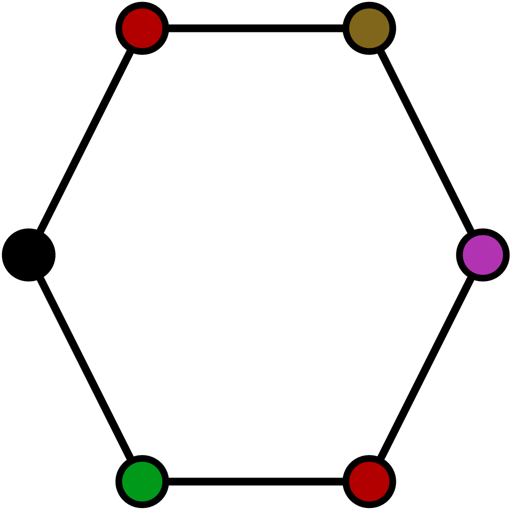
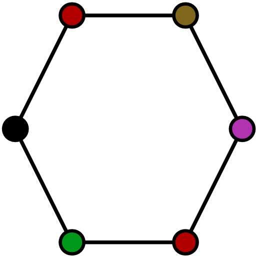


The distinguishing number

Definition

A coloring is called distinguishing, if it is not preserved by any non-trivial automorphism.



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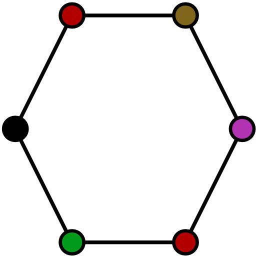

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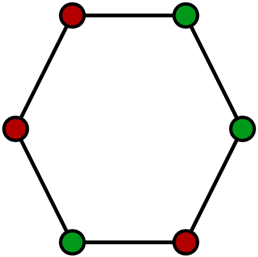

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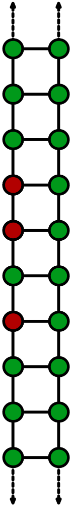

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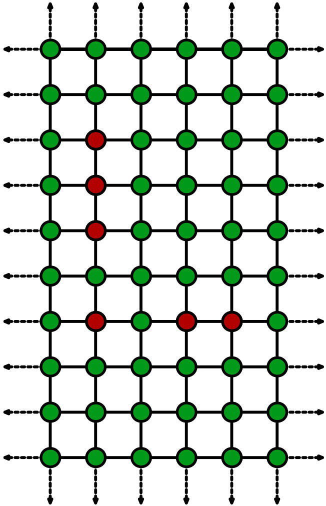

The motion of a graph

Definition

G has motion m if every $\varphi \in \operatorname{Aut} G \setminus \{\mathrm{id}\}$ moves at least m vertices.

Motion and distinguishing number

Lemma (Russel and Sundaram '98)

Let G is a finite graph with motion m and assume that $|\operatorname{Aut} G| \leq 2^{\frac{m}{2}}$ Then G is 2-distinguishable.

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Proof.

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$$\mathbb{P}(c \text{ not distinguishing}) \leq \sum_{\mathrm{id} \neq \varphi \in \operatorname{Aut} G} \mathbb{P}(\varphi \text{ preserves } c) \leq (2^{\frac{m}{2}} - 1)\, 2^{-\frac{m}{2}}$$

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Conjecture (Tucker '11)

If G is a connected locally finite graph and m is infinite, then G is 2-distinguishable.

Some Examples

Tucker's conjecture is true in each of the following cases:

- G is a tree (or at least a "tree like graph")
  (Watkins, Zhou '07; Imrich, Klavžar, Trofimov '07)
- Aut G is countable
  (Imrich et al. '11)
- G satisfies the "distinct spheres condition"
  (Smith, Tucker, Watkins '11)
- G is a cartesian product with at least 2 infinite factors
  (Smith, Tucker, Watkins '11)
- G does not grow "too fast"
  (Cuno, Imrich, L. '12)

Random colourings

We want to:

- Color every vertex with a colour in $\{0, 1\}$ uniformly at random.
- Colours of disjoint vertex sets are independent of each other.

There is a probability measure $\mathbb{P}$ on $\{0, 1\}^{|V|}$ with these properties.

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Theorem (L. '12)

Let G be a graph with infinite motion and countable automorphism group. Then a random coloring is almost surely distinguishing.

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Proof.

$$\mathbb{P}(\exists \varphi \in \mathrm{Aut}(G) \mid \varphi \text{ fixes } c) \leq \sum_{\mathrm{id} \neq \varphi \in \mathrm{Aut}(G)} \mathbb{P}(\varphi \text{ fixes } c) = 0 \qquad \square$$

If Aut G is uncountable...

Aut(G) acts on the set of colourings (from the right) by $c\varphi = c \circ \varphi$.

Clear from the definitions:

$$c \text{ is distinguishing} \Leftrightarrow (\text{Aut } G)_c = \{\text{id}\}$$

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Clear from the definitions:

$$c \text{ is distinguishing} \Leftrightarrow (\mathrm{Aut}\, G)_c = \{\mathrm{id}\}$$

$$c \text{ is ``almost'' distinguishing} \Leftrightarrow (\mathrm{Aut}\, G)_c \text{ is sparse}$$

Two types of sparsity

Theorem (L. '13)

If G has infinite motion then the stabiliser of a random colouring is almost surely closed and nowhere dense in the permutation topology on on Aut G.
Furthermore it is almost surely is a null set with respect to the Haar measure on Aut G.

The permutation topology

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The permutation topology on Aut G is the topology of pointwise convergence, where V is endowed with the discrete topology.

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Aut G with this topology is

- separable
- locally compact
- σ-compact

The Haar measure

Aut G is locally compact $\Rightarrow$ there is a Haar measure $\mathbb{H}$

Aut G is σ-compact $\Rightarrow$ $\mathbb{H}$ is σ-finite

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Theorem (Fubini)

If ν and μ are σ-finite measures and $f \geq 0$ is measurable with respect to the product measure,then

$$\int\int f \, d\nu \, d\mu = \int\int f \, d\mu \, d\nu.$$

The theorem again

Theorem (L. '13)

If G has infinite motion then the stabiliser of a random colouring is almost surely closed and nowhere dense in the permutation topology on on $\operatorname{Aut} G$.
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Aut G is separable $\Rightarrow$ stabiliser of a random colouring is a.s. nowhere dense.

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$$\mathbb{E}(\mathbb{H}((\text{Aut } G)_c)) = \int_{\{0,1\}^{|V|}} \int_{\text{Aut } G} I_{[c\varphi = c]} \, d\mathbb{H}(\varphi) \, d\mathbb{P}(c)$$

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Sparsity with respect to the Haar measure:

$$\begin{aligned}\mathbb{E}(\mathbb{H}((\text{Aut } G)_c)) &= \int_{\{0,1\}^{|V|}} \int_{\text{Aut } G} I_{[c\varphi = c]} \, d\mathbb{H}(\varphi) \, d\mathbb{P}(c) \\ &= \int_{\text{Aut } G} \int_{\{0,1\}^{|V|}} I_{[c\varphi = c]} \, d\mathbb{P}(c) \, d\mathbb{H}(\varphi)\end{aligned}$$

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&= \int_{\text{Aut } G} \int_{\{0,1\}^{|V|}} I_{[c\varphi=c]} \, d\mathbb{P}(c) \, d\mathbb{H}(\varphi) \\
&= \int_{\text{Aut } G} 0 \, d\mathbb{H}(\varphi) \\
&= 0
\end{aligned}$$

A stronger conjecture

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If G is a connected locally finite graph with infinite motion, then a random 2-colouring is almost surely distinguishing.

- for this conjecture it is vital, that G is locally finite
- all of the presented results also work in the more general case of subdegree finite, closed permutation groups
- it suffices to consider compact groups