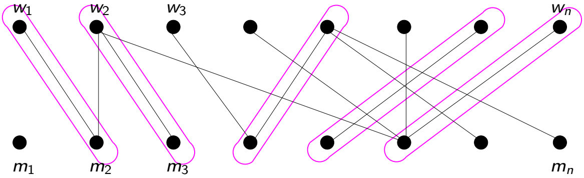
Towards the distribution of the size of the largest non-crossing matchings in random bipartite graphs

Marcos KIWI, U. Chile

joint work with Martin LOEBL, Charles U.

Problem

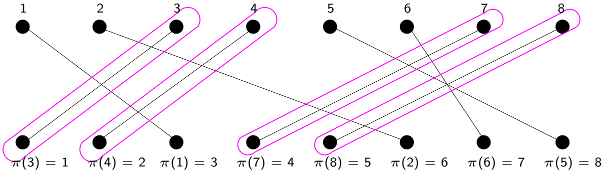


$G \sim \mathcal{G}$

$L(G) = 5$

Instance 1: Longest Increasing Sequence (LIS) Problem

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & \color{magenta}{1} & \color{magenta}{2} & 8 & 7 & \color{magenta}{4} & \color{magenta}{5} \end{pmatrix}$$

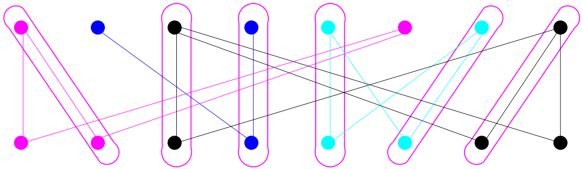


$L(G_\pi) = 4$

Instance 2: Longest Common Subsequence (LCS) Problem

$\alpha = a \ b \ c \ b \ d \ a \ d \ c$

$\beta = a \ a \ c \ b \ d \ d \ c \ c$



$L(G_{\alpha,\beta}) = 6$

Known Results

- LIS model

  Baik-Deift-Johanson - J. AMS'99

  $\frac{L-2\sqrt{n}}{n^{1/6}}$ asymptotically, is Tracy-Widom.

- LCS model

  Loebl-K.-Matousek - AIM'05

  $$\exists \gamma_k = \lim_{n \to +\infty} \frac{1}{n} \mathbf{E}\left(L\right) \text{ and } \gamma_k \sqrt{k} \to 2 \text{ as } k \to \infty.$$

- r-regular graph model

  Loebl-K. - RS&A'02

  $\frac{1}{\sqrt{rn}} \mathbf{E}\left(L\right) \to 2$ as $n \to \infty$ when $r = o(n^{1/4})$.

- Erdös-Renyi model

  Seppäläinen - Ann. App. Prob.'97

  $\frac{1}{n} L \to \frac{2}{1+1/\sqrt{p}}$ a.s. as $n \to \infty$ when $0 < p < 1$.

Focus of this talk

- We consider the uniform distribution over r-regular bipartite multigraphs with n nodes per color class.
- We try to derive/characterize the distribution of L, ... not only its expectation.

Gessel's Identity

If $g_1(n; d)$ denotes the number of permutations of $[n]$ with LIS at most d, and

$$I_\nu(x) = \sum_{m\in\mathbb{N}} \frac{1}{m!(m+\nu)!} \left(\frac{x}{2}\right)^{2m+\nu} ,$$

it holds that

$$\sum_{n\in\mathbb{N}} \frac{g_1(n; d)}{n!^2} x^{2n} = \det\left(I_{|r-s|}(2x)\right)_{1\le r,s\le d} .$$

Equivalently, for $N(t)$ Poisson of rate t and π uniform over $S_{N(t)}$,

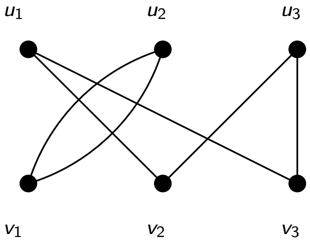
$$\mathbf{Pr}\left[L(G_\pi) \le d\right] = \sum_{n\ge 0} \frac{e^{-t} t^n}{n!} \cdot \frac{g_1(n; d)}{n!} = e^{-t} \det\left(I_{|r-s|}(2\sqrt{t})\right)_{1\le r,s\le d} .$$

Goal

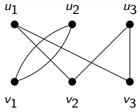
Derive a similar relation but for r-regular bipartite multigraphs

Step 1: Starting point

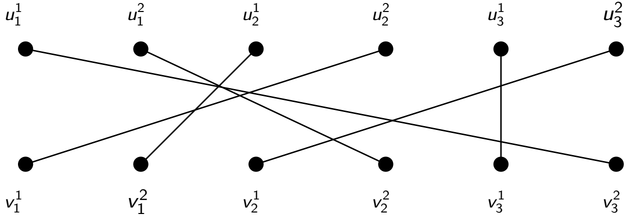
Consider a r-regular n-node per color class bipartite multigraph G such that $L(G) \leq d$



Step 2: Obtain an associated permutation



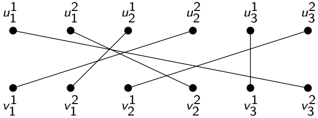
... consider the following permutation π of $[rn]$



Note that:

- π belongs to a restricted class of permutations, and
- $LIS(\pi) \leq d$.

Step 3: Obtain an associated pair of Young Tableaux



... apply the RSK algorithm to π

P

1	3
2	5
4	
6	

Q

1	5
2	6
3	
4	

Note that (P, Q) belongs to a restricted class of equal λ-shape Young tableaux, where λ is a partition of rn.

Step 4: Obtain an associated lattice walk

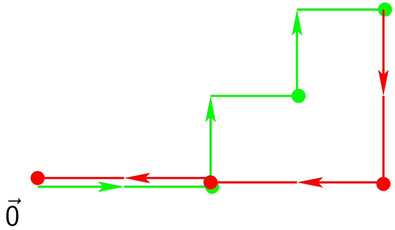
P

1	3
2	5
4	
6	

Q

1	5
2	6
3	
4	

... consider the following walk ω in $\mathbb{Z}^d$



Note that ω belongs to a restricted class of $2rn$ step closed walks.

Summarizing

We want to determine exactly the number of walks in $\mathbb{Z}^d$ that:

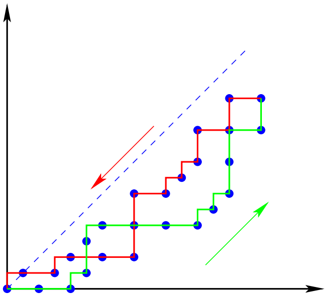
(i) Start at $\vec{0}$.

(ii) Steps in positive direction come before steps in negative direction.

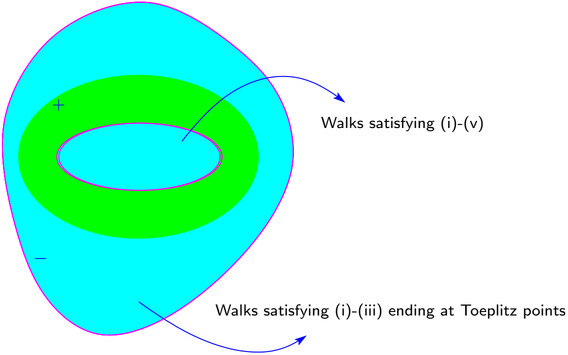
(iii) The i-th block of r steps (i.e. steps $ir+1, \ldots, (i+1)r$) are in non-increasing order of dimension.

(iv) End at $\vec{0}$.

(v) Stay in the region $x_1 \geq x_2 \geq \ldots \geq x_d$.



Step 5: Define a parity reversing involution



Conclusion

Let

- $W(d, r, 2rn; T(\pi))$ denote the $2rn$-step walks in $\mathbb{Z}^d$ satisfying (i)-(iii) and ending at Toeplitz point $T(\pi)$
- $g_r(n; d)$ denote the number of r-regular n node per color class bipartite multi-graphs G such that $L(G) \leq d$

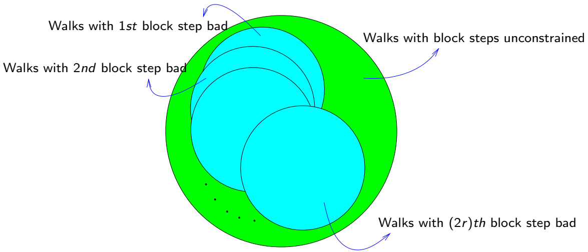
Then,

$$g_r(n; d) = \sum_{\pi \in S_d} \operatorname{sign}(\pi) \left| W(d, r, 2rn; T(\pi)) \right| .$$

Consequences

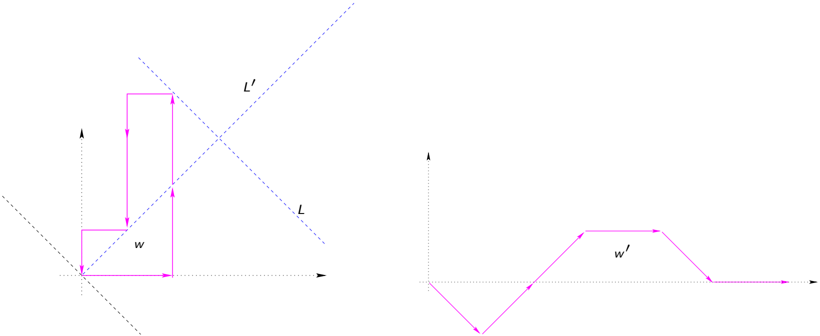
.

Applying the Inclusion-Exclusion principle:



... and obtain $g_r(n; d)$ for some small values of r and n.

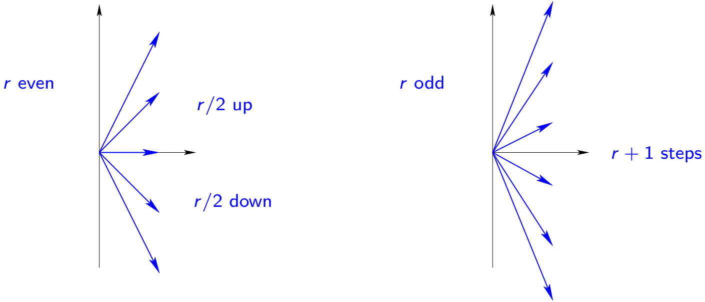
Symmetrization technique (e.g. $r = d = 2$ case)



Induced mapping

From $w \in W(d = 2, r, 2rn; T(\pi))$ to walks

- With steps:



- Ending at: $(2rn, 0)$ if $\pi = id$, and $(2rn, 2)$ if $\pi = (12)$,

Kernel Method (case r odd)

Consider the Laurent polynomial $P_r(u) = \frac{1}{u^r} + \frac{1}{u^{r-1}} + \ldots + u^{r-1} + u^r$.

Note that

$$|W(2, r, 2rn; (0,0))| = [z^{2n}] \sum_{n \in \mathbb{N}} (zP_r(u))^n = [z^{2n}] \frac{1}{1 - zP_r(u)},$$

$$|W(2, r, 2rn; (-1,1))| = [z^{2n}u^2] \sum_{n \in \mathbb{N}} (zP_r(u))^n = [z^{2n}u^2] \frac{1}{1 - zP_r(u)}.$$

Thus (see Banderier & Flajolet - TCS'02):

$$g_r(n; 2) = [z^{2n-1}] G_{r,2}(z) = [z^{2n-1}] \sum_{j=1}^{r} u_j'(z) \left(\frac{1}{u_j(z)} - u_j(z) \right),$$

where $u_1, \ldots, u_r$ are the "small branches" of the *characteristic equation* $u^r - zu^r P_r(u) = 0$.

Consequences ($d = 2$ case)

$\underline{r = 1}$: One small branch $u_1(z) = \frac{1}{2z}(1-\sqrt{1-4z^2})$ of the characteristic equation with $P_1(u) = u^{-1} + u$.

$$G_{1,2}(z) = \frac{1-\sqrt{1-4z}}{2z^2} = 1 + z^2 + 2z^4 + 5z^6 + 14z^8 + 42z^{10} + 132z^{12} + 429z^{14} + 1430z^{16} + 4862z^{18} \ldots$$

$\underline{r = 2}$: One small branch $u_1(z) = \frac{1}{2z}(1-z-\sqrt{1-2z-3z^2})$ of the characteristic equation with $P_2(u) = u^{-1} + 1 + u$.

$$G_{2,2}(z) = \frac{1+z-\sqrt{1-2z-3z^2}}{2z(1+z)} = 1 + z^2 + z^3 + 3z^4 + 6z^5 + 15z^6 + 36z^7 + 91z^8 + 232z^9 + 603z^{10} \ldots$$

$\underline{r = 3}$: $G_{3,2}(z) = 1 + z^2 + 4z^4 + 34z^6 + 364z^8 + 4269z^{10} + 52844z^{12} + 679172z^{14} + 8976188z^{16} \ldots$

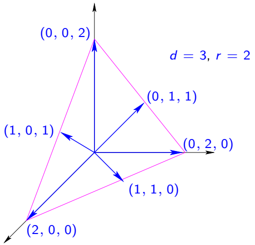
$\underline{r = 4}$: $G_{4,2}(z) = 1 + z^2 + z^3 + 5z^4 + 16z^5 + 65z^6 + 260z^7 + 1085z^8 + 4600z^9 + 19845z^{10} \ldots$

Related to EIS **A000108**, **A005043**, and **A007043**.

To conclude ...

Is there a generating function approach for the $d > 2$ case?

Steps are now $\mathbb{Z}^d$ vectors satisfying $x_1 + \ldots + x_d = \pm r$.



- If only positive steps are taken, how many n step walks starting at $\vec{0}$ end in $nr\vec{\mathbb{I}}$?
- What about the number of signed sums of walks ending in Toeplitz points?

THE END!