Bi-angular lines in $\mathbb{R}^n$

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- Mutually unbiased weighing matrices

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Any set of MU Hadamard matrices of order n forms a set of bi-angular lines in $\mathbb{R}^n$ with $\alpha = \frac{1}{\sqrt{n}}$.

Mutually unbiased weighing matrices

Definition

A matrix $W = [w_{ij}]$ of order n and $w_{ij} \in \{-1, 0, 1\}$ is called a *weighing matrix with weight p* if $WW^t = pI_n$, where I_n is the identity matrix of order n.

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The DGS upper bound

Theorem: Let m be the number of bi-angular lines in $\mathbb{R}^n$.

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$$m \leq \begin{cases} \dfrac{n(n+2)(1-\alpha^2)}{3-(n+2)\alpha^2} & \text{if } \; 3-(n+2)\alpha^2 > 0, \\ \dfrac{n(n+1)(n+2)}{6} & \text{otherwise.} \end{cases}$$

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Theorem: Let m' be the number of MU Hadamard matrices of order n. Then

$$m' \leq \frac{n}{2}.$$

The two upper bounds differ by one for $n = 4k^2$, the order of a Hadamard matrix ($\alpha = \frac{1}{2k}$).

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4	12	12

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5	21	20

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5	21	20
6	36	30
7	63	63
8	120	120
9	165	120

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Type	DGS UB	Number Found
W(4,4)	2	2

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W(6,4)	5	4
W(7,4)	8	8
W(8,4)	14	14

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The perpendicularity graph of the Gram matrix of the 63 vectors is an SRG(63,30,13,15).The vertices are disjoint union of 9 cliques of size 7. The graph is isomorphic to the classical design having as blocks the hyperplanes in PG(5,2).

Mutually suitable Latin squares

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There are $p-1$ MSLS of size p for each prime power p.

The auxiliary matrices corresponding to weighing matrices

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Theorem

There is a weighing matrix $W(n,p)$ W *of order n and weight p if and only if there are n auxiliary* $(0,\pm1)$- *matrices* $C_0, C_1, C_2, \ldots, C_{n-1}$ *of order n such that:*

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Proof.

Let r_i be the $i+1$-th row of W and
let $C_i = r_i^t r_i$, $i = 0, 1, \ldots, n-1$. □

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Lemma: If there is a $W(n,p)$ and q MSLS of size m, $m \geq n$. Then there are q mutually unbiased weighing matrices (MUWM), $W(nm, p^2)$.

An example of MU weighing matrices

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$$\text{Let } W = \begin{pmatrix} 0 & 1 & 1 & 1 \\ - & 0 & 1 & - \\ - & - & 0 & 1 \\ - & 1 & - & 0 \end{pmatrix}.$$

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$$C_0 = r_0^t r_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad C_1 = r_1^t r_1 = \begin{pmatrix} 1 & 0 & - & 1 \\ 0 & 0 & 0 & 0 \\ - & 0 & 1 & - \\ 1 & 0 & - & 1 \end{pmatrix},$$

$$C_2 = r_2^t r_2 = \begin{pmatrix} 1 & 1 & 0 & - \\ 1 & 1 & 0 & - \\ 0 & 0 & 0 & 0 \\ - & - & 0 & 1 \end{pmatrix}, \quad C_3 = r_3^t r_3 = \begin{pmatrix} 1 & - & 1 & 0 \\ - & 1 & - & 0 \\ 1 & - & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$W_1 = \begin{pmatrix} C_0 & C_3 & C_1 & 0 & C_2 \\ C_2 & C_0 & C_3 & C_1 & 0 \\ 0 & C_2 & C_0 & C_3 & C_1 \\ C_1 & 0 & C_2 & C_0 & C_3 \\ C_3 & C_1 & 0 & C_2 & C_0 \end{pmatrix} \quad W_2 = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 & 0 \\ 0 & C_0 & C_1 & C_2 & C_3 \\ C_3 & 0 & C_0 & C_1 & C_2 \\ C_2 & C_3 & 0 & C_0 & C_1 \\ C_1 & C_2 & C_3 & 0 & C_0 \end{pmatrix},$$

$$W_3 = \begin{pmatrix} C_0 & C_2 & 0 & C_1 & C_3 \\ C_3 & C_0 & C_2 & 0 & C_1 \\ C_1 & C_3 & C_0 & C_2 & 0 \\ 0 & C_1 & C_3 & C_0 & C_2 \\ C_2 & 0 & C_1 & C_3 & C_0 \end{pmatrix},$$

$$W_4 = \begin{pmatrix} C_0 & 0 & C_3 & C_2 & C_1 \\ C_1 & C_0 & 0 & C_3 & C_2 \\ C_2 & C_1 & C_0 & 0 & C_3 \\ C_3 & C_2 & C_1 & C_0 & 0 \\ 0 & C_3 & C_2 & C_1 & C_0 \end{pmatrix}.$$

W_1, W_2, W_3, W_4 form a set of four MUWM of order 20 and weight 9.

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Lemma: If there is a $W(n,n)$ and q MSLS of size m on the set $\{0,1,2,\ldots,m-1\}$, $m \geq n-1$. Then there are mnq biangular lines in $\mathbb{R}^{mn}$.

Biangular lines and association schemes

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- Next page for more association schemes.

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- $H_{12} + MSLS(11)$

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- $H_{12} + MSLS(11) \longrightarrow AS(1452; 600, 600, 120, 120, 6, 5)$

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- $H_{12} + MSLS(11) \longrightarrow AS(1452; 600, 600, 120, 120, 6, 5)$
- $H_{20} + MSLS(19)$

This is the next page!

(... *Details Omitted* ...)

From this construction, we were able to use small orders of Hadamard matrices and MSLS to generate large 6-association schemes.

- $H_{12} + MSLS(11) \longrightarrow AS(1452; 600, 600, 120, 120, 6, 5)$
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For example, MU Hadamard matrices of order 4^n can be used to generate Kerdock codes. However, in practice the reverse is done!

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  BCH codes of length 128 and distance in $\{56, 64, 72\}$ may be of help.

Thank you organizers!

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Thank you Ian!