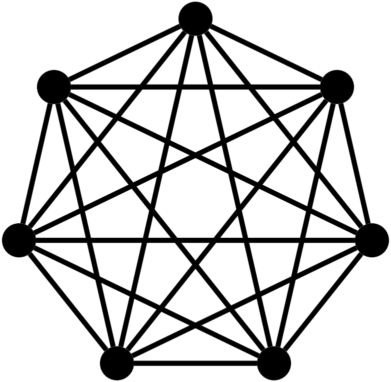
Embedding partial Steiner triple systems with few triples

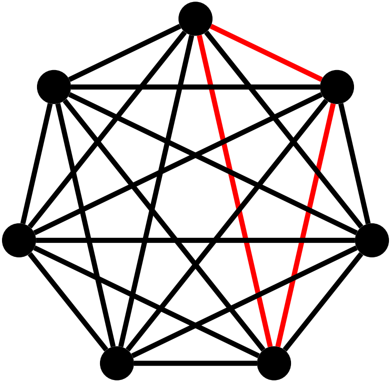
Daniel Horsley (Monash University, Australia)

Steiner triple systems

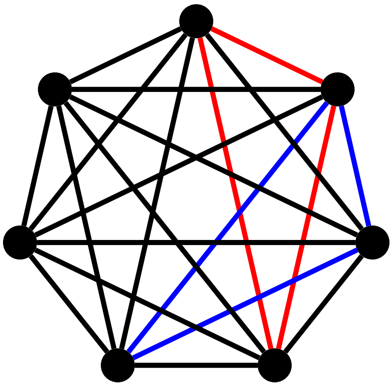
Steiner triple systems



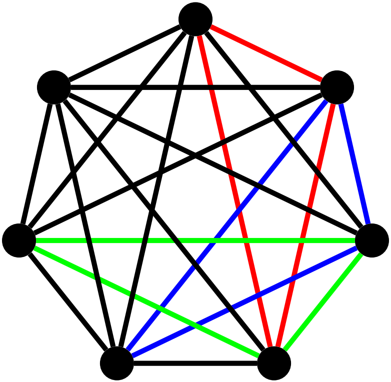
Steiner triple systems



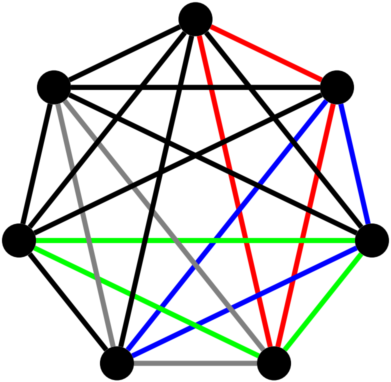
Steiner triple systems



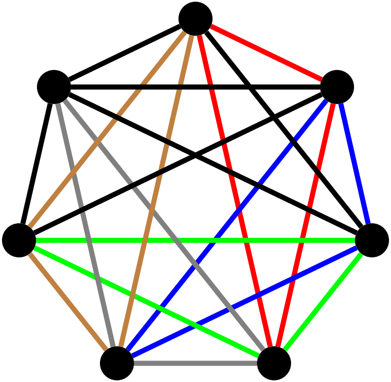
Steiner triple systems



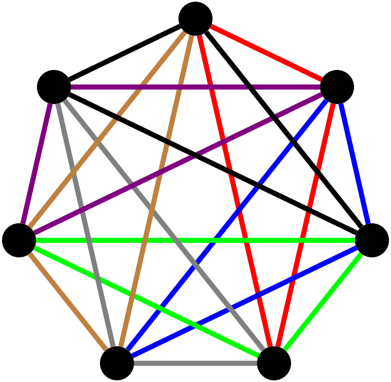
Steiner triple systems



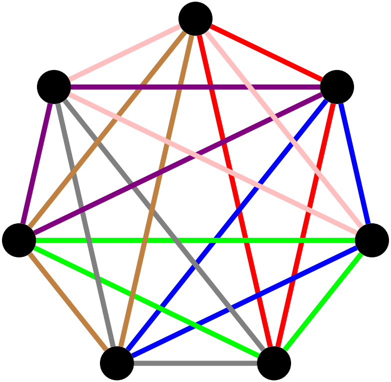
Steiner triple systems



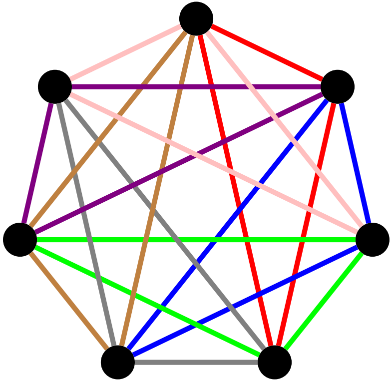
Steiner triple systems



Steiner triple systems

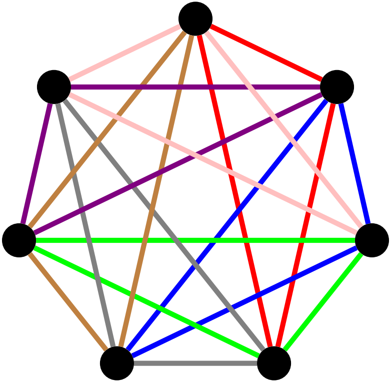


Steiner triple systems



An STS(7)

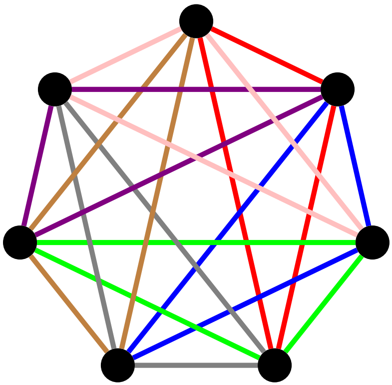
Steiner triple systems



An STS(7)

Theorem (Kirkman 1847) An STS(v) exists if and only if $v \geq 1$ and $v \equiv 1, 3 \pmod{6}$.

Steiner triple systems



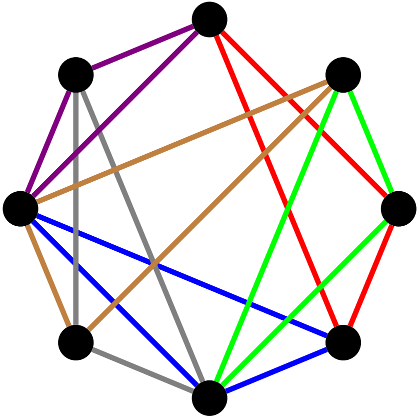
An STS(7)

Theorem (Kirkman 1847) An STS(v) exists if and only if $v \geq 1$ and $v \equiv 1, 3 \pmod 6$.

Call these integers *admissible*.

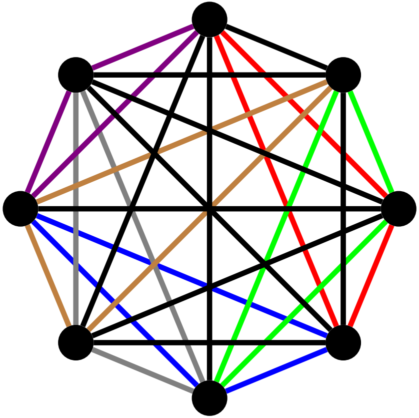
Partial Steiner triple systems

Partial Steiner triple systems



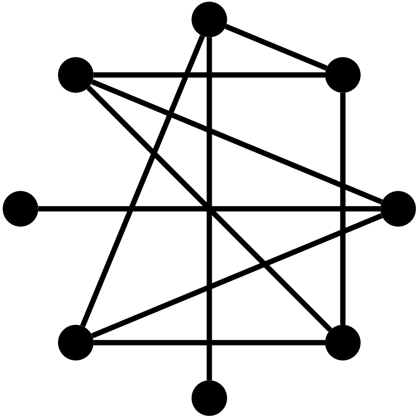
A PSTS(8)

Partial Steiner triple systems



A PSTS(8)

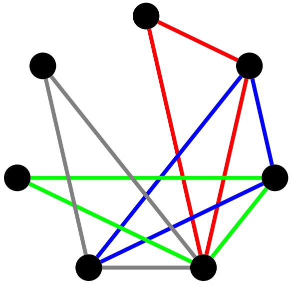
Partial Steiner triple systems



The leave of the PSTS(8)

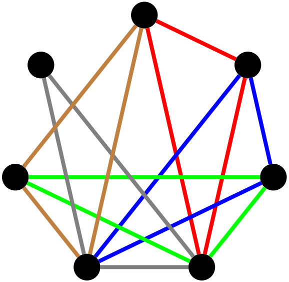
Completing partial Steiner triple systems

Completing partial Steiner triple systems



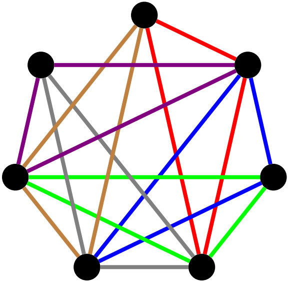
A PSTS(7)

Completing partial Steiner triple systems



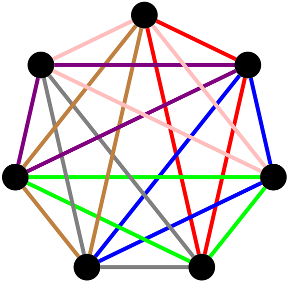
A PSTS(7)

Completing partial Steiner triple systems



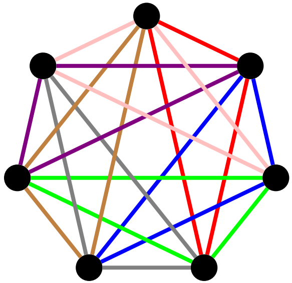
A PSTS(7)

Completing partial Steiner triple systems



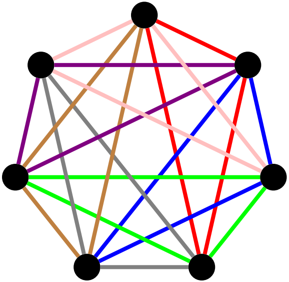
A PSTS(7)

Completing partial Steiner triple systems

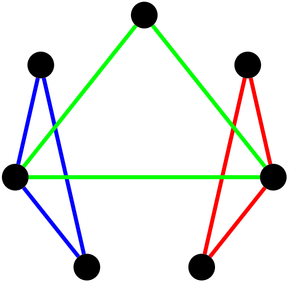


A completion of the PSTS(7)

Completing partial Steiner triple systems

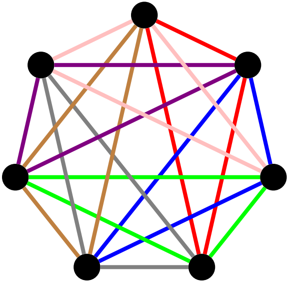


A completion of the PSTS(7)

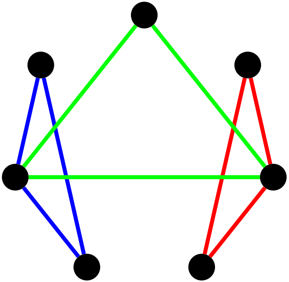


A PSTS(7)

Completing partial Steiner triple systems



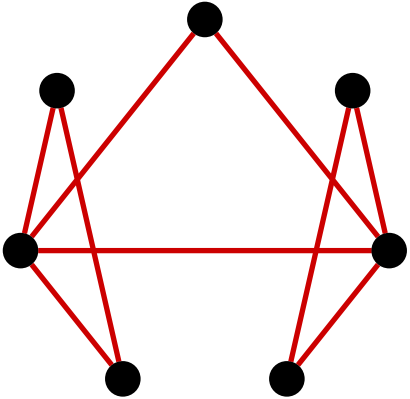
A completion of the PSTS(7)



A PSTS(7) with no completion

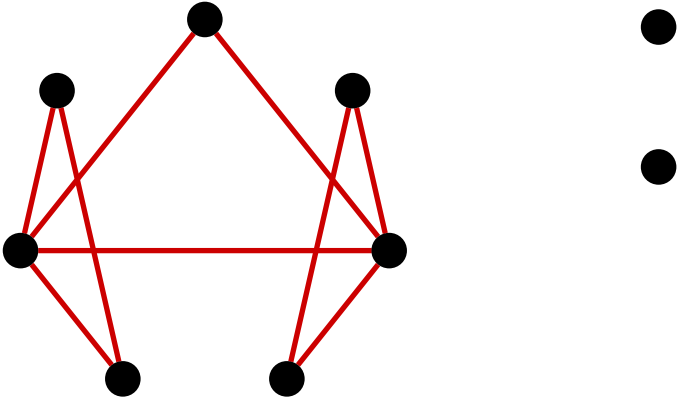
Embeddings of partial Steiner triple systems

Embeddings of partial Steiner triple systems



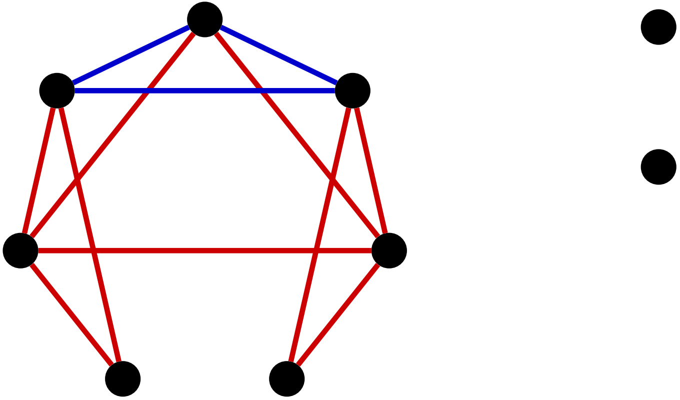
A PSTS(7)

Embeddings of partial Steiner triple systems



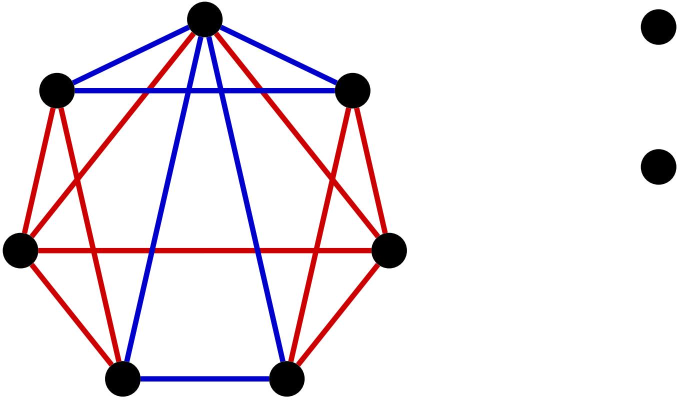
A PSTS(7)

Embeddings of partial Steiner triple systems



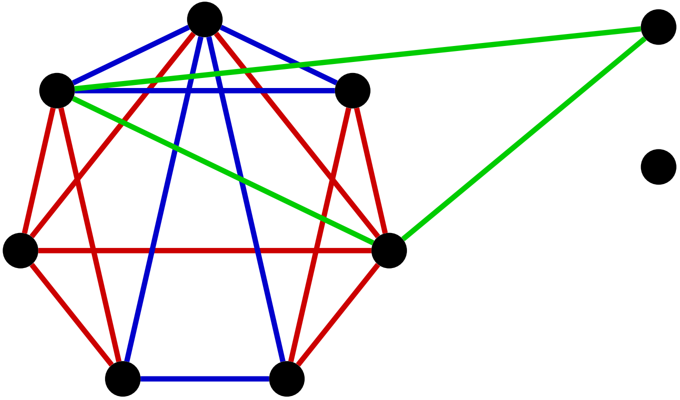
A PSTS(7)

Embeddings of partial Steiner triple systems



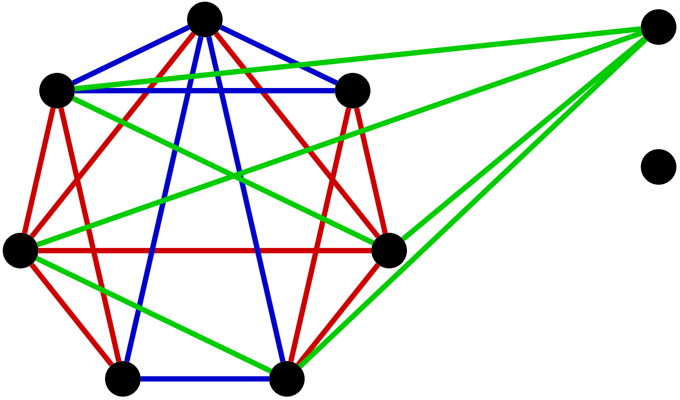
A PSTS(7)

Embeddings of partial Steiner triple systems



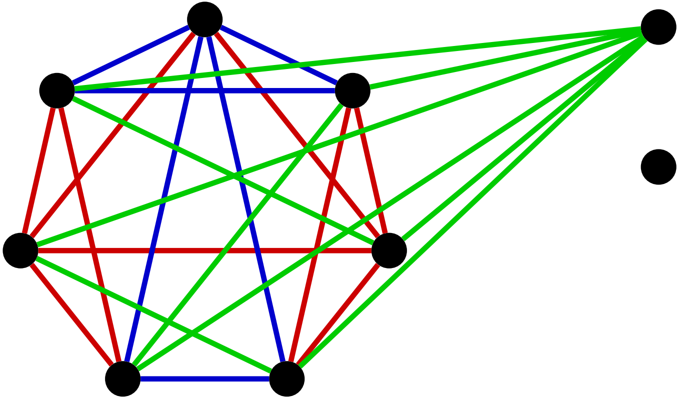
A PSTS(7)

Embeddings of partial Steiner triple systems



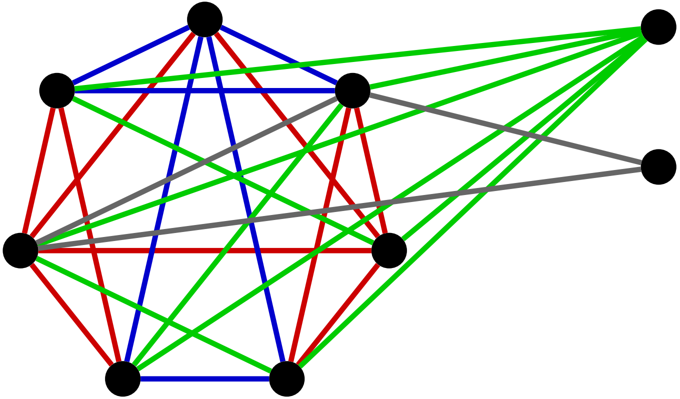
A PSTS(7)

Embeddings of partial Steiner triple systems



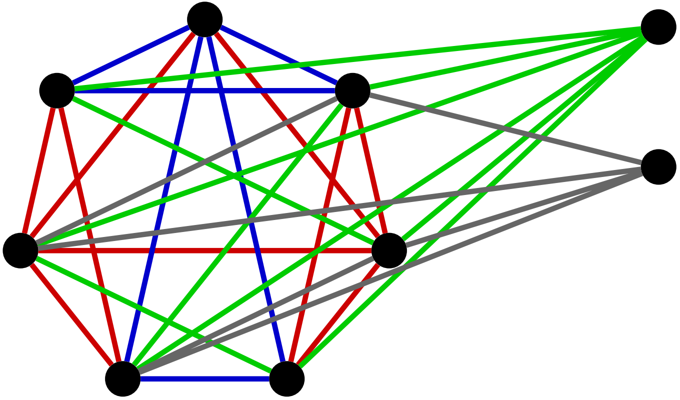
A PSTS(7)

Embeddings of partial Steiner triple systems



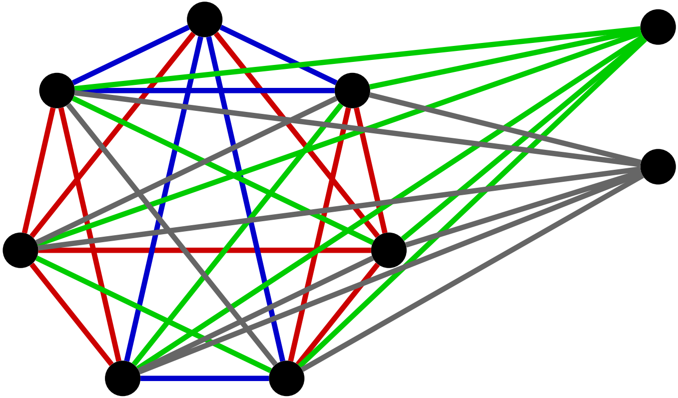
A PSTS(7)

Embeddings of partial Steiner triple systems



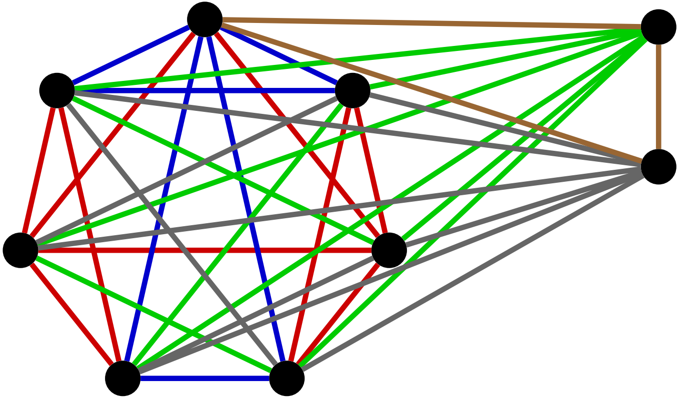
A PSTS(7)

Embeddings of partial Steiner triple systems



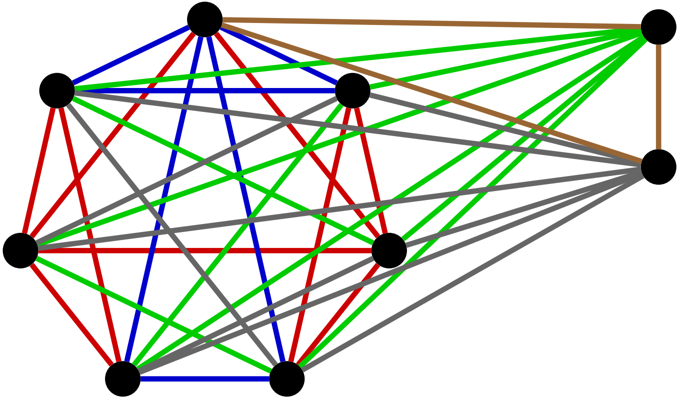
A PSTS(7)

Embeddings of partial Steiner triple systems



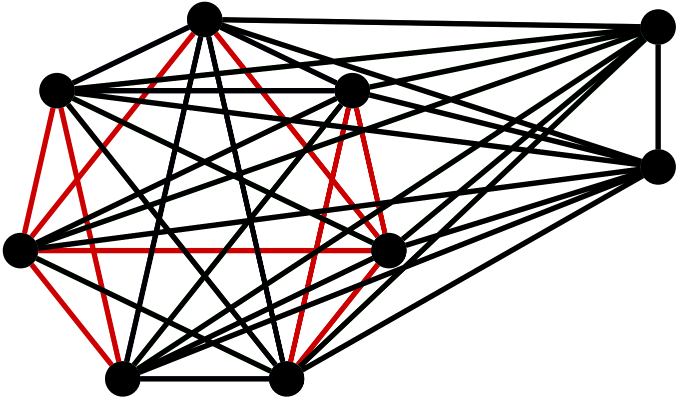
A PSTS(7)

Embeddings of partial Steiner triple systems



An embedding of the PSTS(7) of order 9

Embeddings of partial Steiner triple systems



$L \vee K_2$ was decomposed into triangles

Lindner's Conjecture

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u + 1$.

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u + 1$.

Partial results from many people including

- Treash (1971),
- Lindner (1975),
- Andersen, Hilton, Medelsohn (1981),
- Bryant (2004).

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u + 1$.

Partial results from many people including

- Treash (1971),
- Lindner (1975),
- Andersen, Hilton, Medelsohn (1981),
- Bryant (2004).

Theorem (Bryant, Horsley 2009) The conjecture holds.

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u+1$.

Partial results from many people including

- Treash (1971),
- Lindner (1975),
- Andersen, Hilton, Medelsohn (1981),
- Bryant (2004).

Theorem (Bryant, Horsley 2009) The conjecture holds.

Many "dense" PSTS(u)s have no embeddings of order less than $2u+1$.

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u+1$.

Partial results from many people including

- Treash (1971),
- Lindner (1975),
- Andersen, Hilton, Medelsohn (1981),
- Bryant (2004).

Theorem (Bryant, Horsley 2009) The conjecture holds.

Many "dense" PSTS(u)s have no embeddings of order less than $2u+1$. But most PSTSs do have such embeddings.

Lindner's Conjecture

Conjecture (Lindner 1977) Any PSTS(u) has an embedding of order v for each admissible $v \geq 2u+1$.

Partial results from many people including

- Treash (1971),
- Lindner (1975),
- Andersen, Hilton, Medelsohn (1981),
- Bryant (2004).

Theorem (Bryant, Horsley 2009) The conjecture holds.

Many "dense" PSTS(u)s have no embeddings of order less than $2u+1$. But most PSTSs do have such embeddings.

Call embeddings of order less than $2u+1$ *small*.

Results on small embeddings

Results on small embeddings

Colbourn, Colbourn, Rosa (1983) Showed “double stars” are completable.

Bryant (2002) Gave necessary and sufficient conditions for a PSTS with a $\{0, d\}$-regular leave to have an embedding of order $u + d$. Determined the embedding spectrum in the case $d = 2$.

Bryant, Maenhaut, Quinn, Webb (2004) Determined the embedding spectrum for all PSTS(10)s with 3-regular leaves.

Bryant, Horsley (2006) Determined the embedding spectrum for PSTSs with complete bipartite leaves.

Horsley (201?) Found “half” of the possible small embeddings for PSTS(u)s with $\Delta(L) \leq \frac{1}{4}(u - 9)$ and $|E(L)| < \frac{1}{32}(u - 5)(u - 11) + 2$.

Embedding sparse PSTSs

Embedding sparse PSTSs

Conjecture (Nash-Williams 1970) Any even graph G with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3}{4}|V(G)|$ has a decomposition into triangles.

Embedding sparse PSTSs

Conjecture (Nash-Williams 1970) Any even graph G with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3}{4}|V(G)|$ has a decomposition into triangles.

This conjecture would imply that any PSTS(u) with at most $\frac{u-1}{8}$ triples on each vertex has an embedding of order v for each admissible $v \geq u$.

Embedding sparse PSTSs

Conjecture (Nash-Williams 1970) Any even graph G with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3}{4}|V(G)|$ has a decomposition into triangles.

This conjecture would imply that any PSTS(u) with at most $\frac{u-1}{8}$ triples on each vertex has an embedding of order v for each admissible $v \geq u$.

Very little progress has been made on this conjecture.

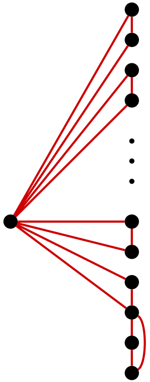
Embedding PSTSs with few triples

Embedding PSTSs with few triples

Not every PSTS of admissible order with few triples is completable.

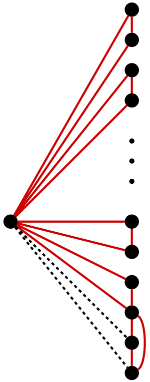
Embedding PSTSs with few triples

Not every PSTS of admissible order with few triples is completable.



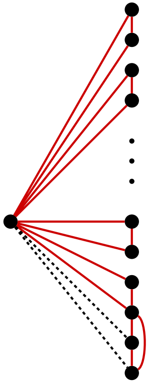
Embedding PSTSs with few triples

Not every PSTS of admissible order with few triples is completable.



Embedding PSTSs with few triples

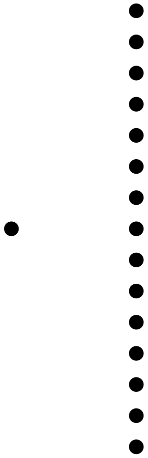
Not every PSTS of admissible order with few triples is completable.



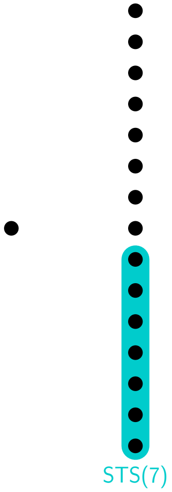
Question Is every PSTS(u) with u admissible and fewer than $\frac{u-1}{2}$ triples completable?

Embedding PSTSs with few triples

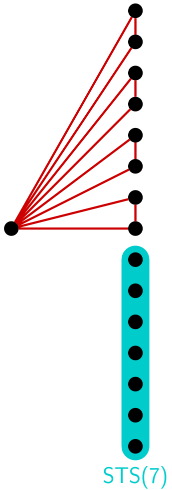
Embedding PSTSs with few triples



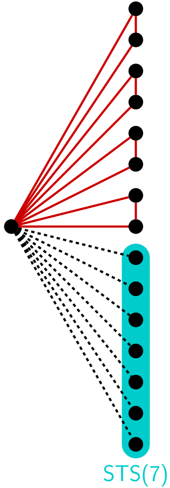
Embedding PSTSs with few triples



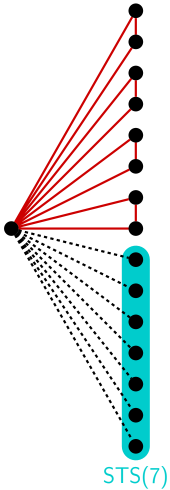
Embedding PSTSs with few triples



Embedding PSTSs with few triples



Embedding PSTSs with few triples



A PSTS(16) with no embedding of order less than 23

Embedding PSTSs with few triples

Embedding PSTSs with few triples

Theorem (Horsley 201?) Any PSTS(u) with $u \geq 87$ and at most $\frac{u^2}{50} - \frac{11u}{100} + \frac{116}{75}$ triples has an embedding for each admissible order $v \geq \frac{8u+17}{5}$.

Embedding PSTSs with few triples

Theorem (Horsley 201?) Any PSTS(u) with $u \geq 87$ and at most $\frac{u^2}{50} - \frac{11u}{100} + \frac{116}{75}$ triples has an embedding for each admissible order $v \geq \frac{8u+17}{5}$.

For sufficiently large u there is a PSTS(u) with at most $\frac{u^2}{50} - \frac{11u}{100} + \frac{116}{75}$ triples that does not have an embedding of order v for any $v < (1.346)u$.

Back to Nash-Williams Conjecture

Back to Nash-Williams Conjecture

Conjecture (Nash-Williams 1970) Any even graph G with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3}{4}|V(G)|$ has a decomposition into triangles.

Back to Nash-Williams Conjecture

Conjecture (Nash-Williams 1970) Any even graph G with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geq \frac{3}{4}|V(G)|$ has a decomposition into triangles.

Theorem (Horsley 201?) Any even graph G of order $v \geq 144$ with at least $\frac{3v+17}{8}$ vertices of degree $v-1$, $|E(G)| \equiv 0 \pmod 3$ and $E(G) \geq \binom{v}{2} - (\frac{3v^2}{128} - \frac{31v}{128} - \frac{1}{2})$ has a decomposition into triangles.

The End