Decomposing complete bipartite graphs into short cycles and related results

Daniel Horsley (Monash University, Australia)

Cycle decompositions of $K_{a,b}$

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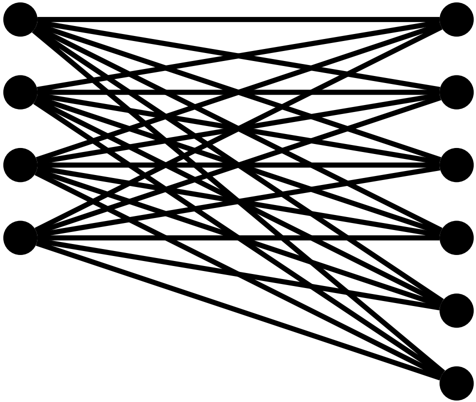

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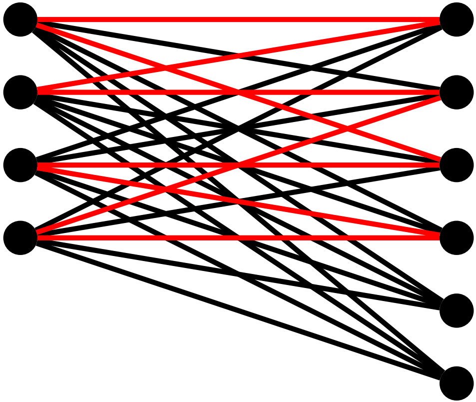

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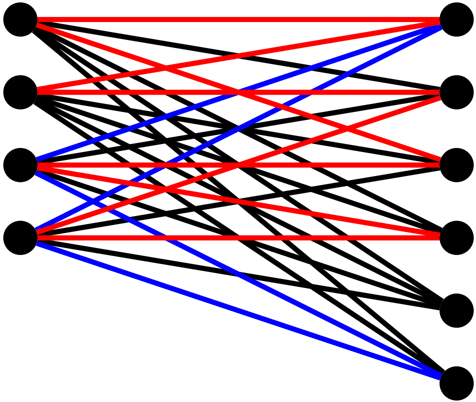

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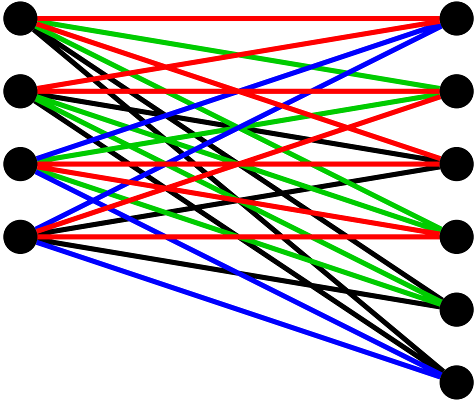

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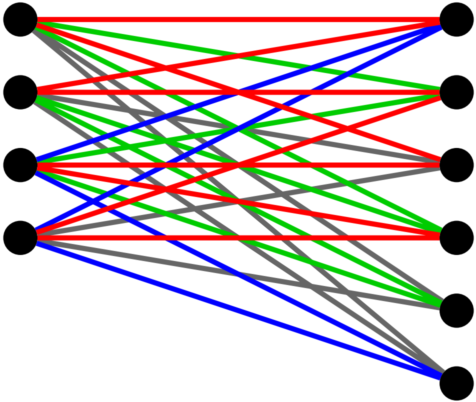

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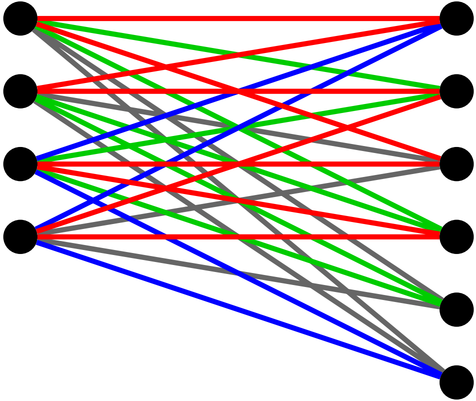

An $(8, 6, 6, 4)$-decomposition of $K_{4,6}$

Cycle decompositions of $K_{a,a} - I$

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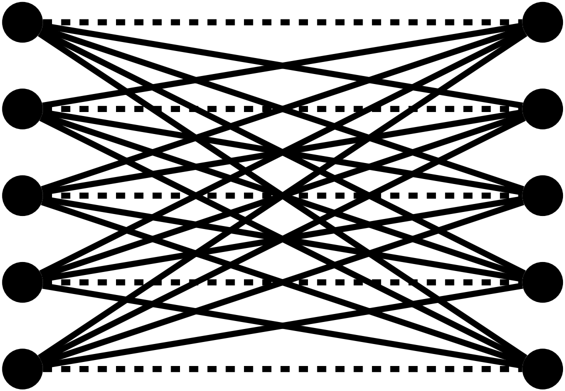

$K_{5,5} - I$

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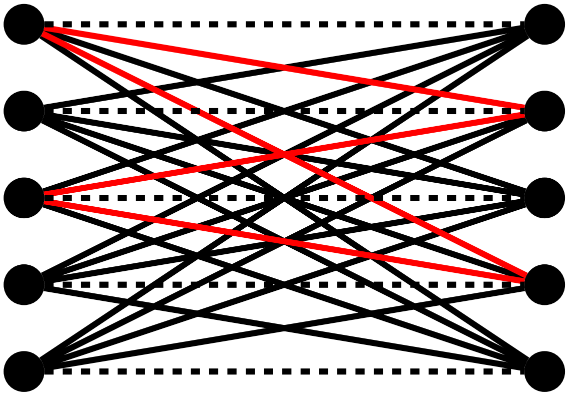

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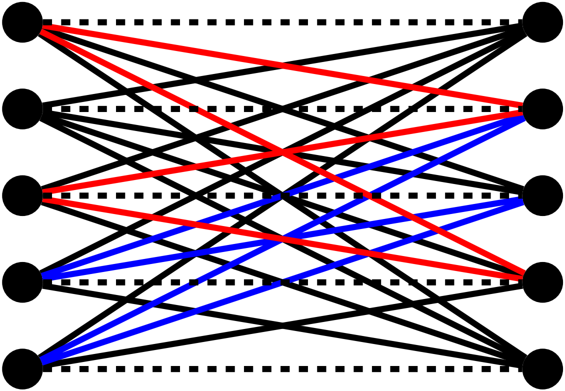

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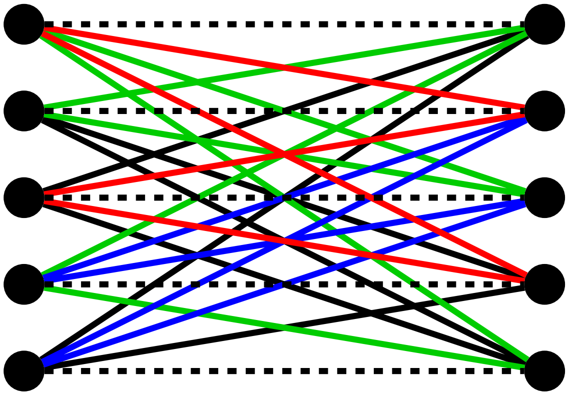

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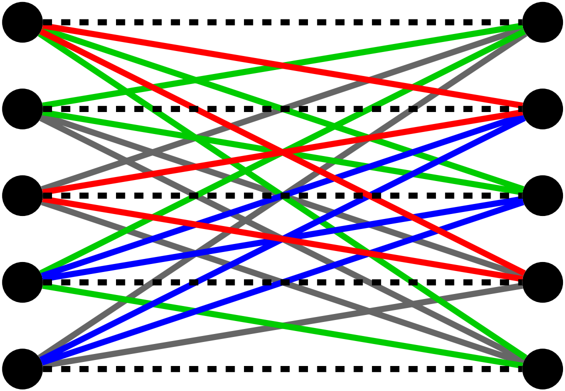

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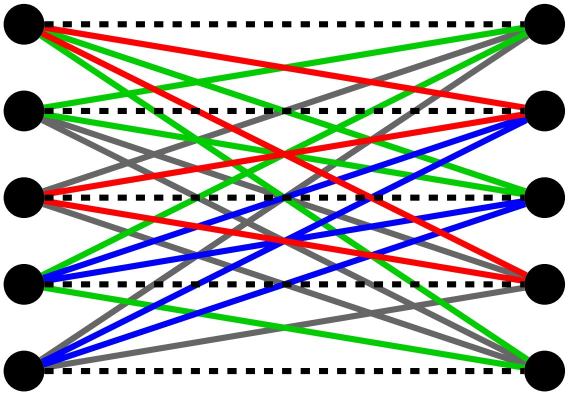

A $(6, 6, 4, 4)$-decomposition of $K_{5,5} - I$

Necessary Conditions

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If there exists an $(m_1, \ldots, m_t)-$decomposition of $K_{a,b}$ then

(1) a and b are even;

(2) $m_1 + \cdots + m_t = ab$;

(3) $m_1, \ldots, m_t \in \{4, 6, 8, \ldots, 2\min(a, b)\}$; and

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If there exists an $(m_1, \ldots, m_t)-$decomposition of $K_{a,a} - I$ then

(1′) a is odd;

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There exists a $(m_1, \ldots, m_t)$-decomposition of $K_{a,b}$ when (1), (2), (3), (4) and any of the following hold.

- $m_1 = \cdots = m_t$ — Sotteau (1981)
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- each $m_i \in \{4, m\}$ for $m \in \{10, 12\}$ — Chou, Fu (2007)

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Lemma If there exists an $(m_1, \ldots, m_t, c, x, y)$-decomposition of $K_{a,b}$, where

(1) $c \geq \frac{1}{3}(x + y)$; and

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- Take the corresponding $(4, \ldots, 4, 6, \ldots, 6)$-decomposition of $K_{a,b}$ (Chou, Fu, Huang).

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- Repeatedly use the join lemma to progressively join the cycles. □

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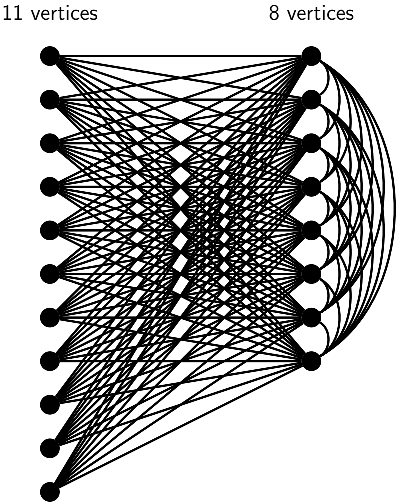

$K_{19} - K_{11}$

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Theorem (Horsley 2012) Let $m \geq 4$ be even. There exists an m-cycle decomposition of $K_v - K_u$ if (1) and (2) hold, $u \geq m+1$ and $v - u \geq m$.

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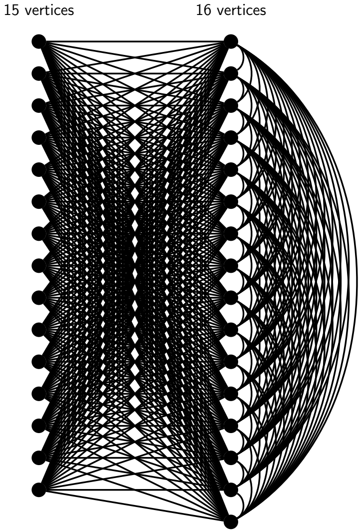

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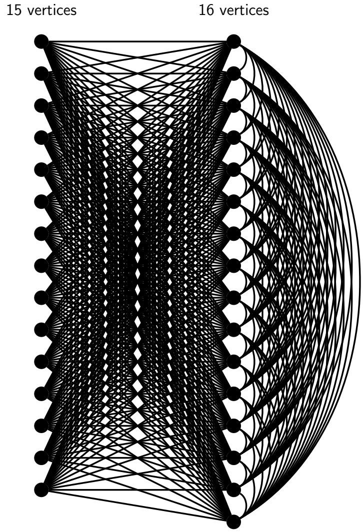
Take a $(10^{22}, 4)$-decomposition of $K_{14,16}$.



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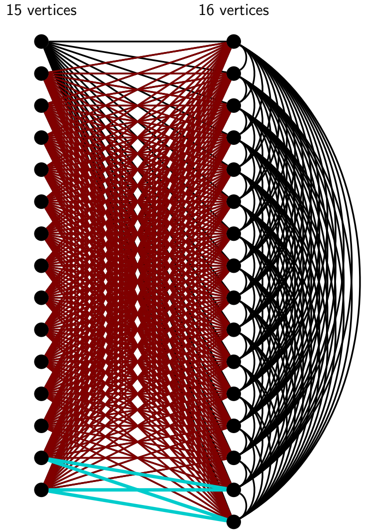

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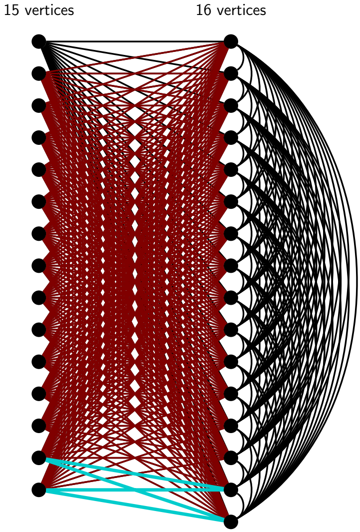

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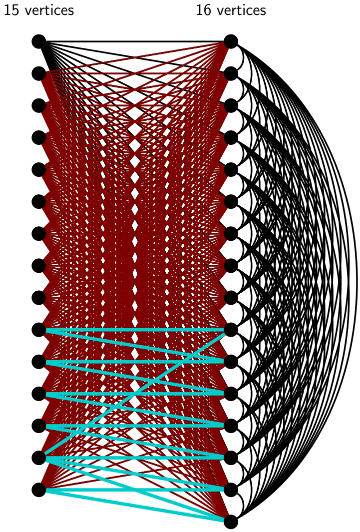

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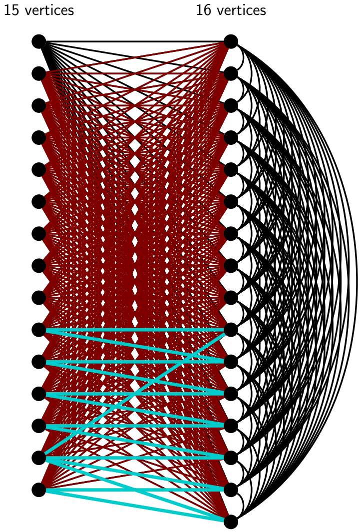
Break these cycles into a 6-path and an 8-path.



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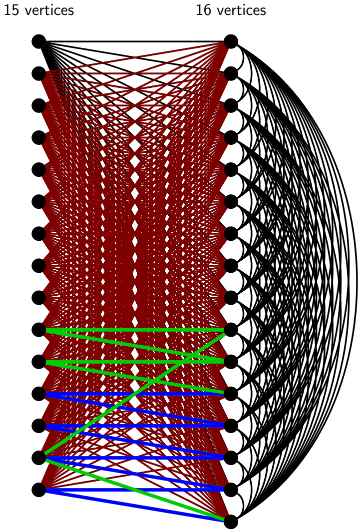

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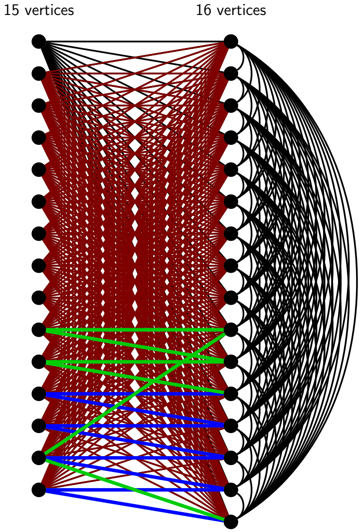

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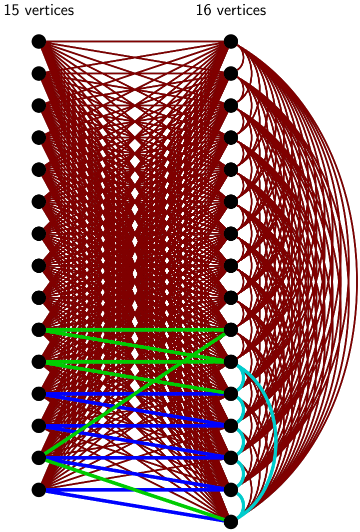

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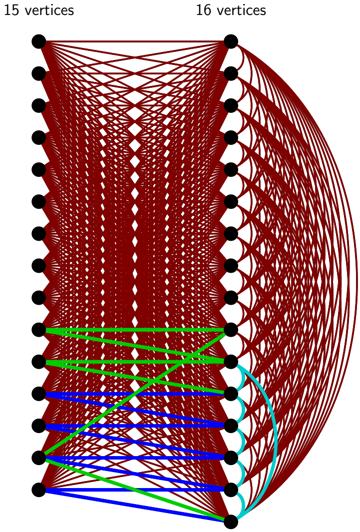
Break the 6-cycle into a 4-path and a 2-path to reconstruct two 10-cycles.



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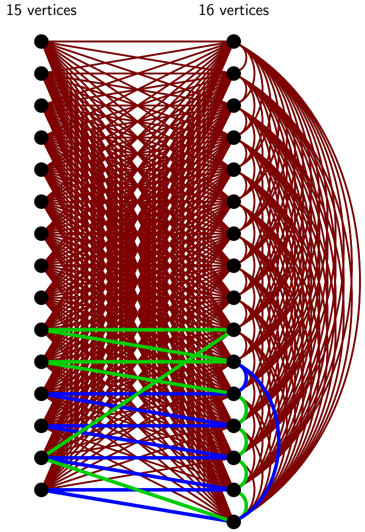
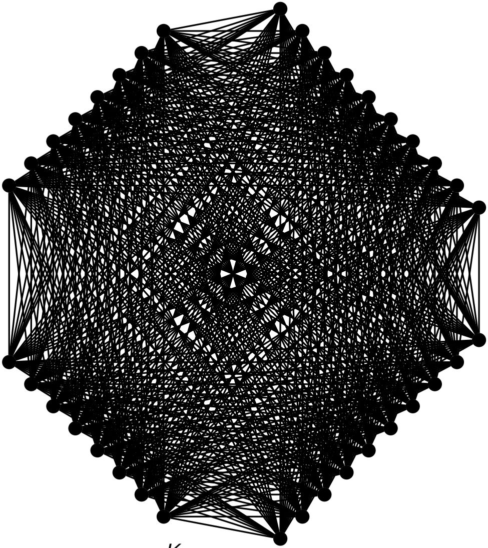


Complete multipartite graphs into m-cycles (m even)

Complete multipartite graphs into *m*-cycles (*m* even)



$K_{8,8,10,10}$

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(3) a_i is even for $i = 1, \ldots, t$ OR a_i is odd for $i = 1, \ldots, t$ and t is odd; and

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(4) $a_t \geq |V(K_{a_1,\dots,a_t})| - \frac{2}{m}|E(K_{a_1,\dots,a_t})|$ (where $a_1 \leq \cdots \leq a_t$).

Theorem (Cavenagh, Billington 2000) Let $m \geq 4$ be even. There exists an m-cycle decomposition of $K_{a_1,\dots,a_t}$ if (1), (2), (3) and (4) hold and $m \in \{4, 6, 8\}$.

Theorem (Horsley 2012) Let $m \geq 4$ be even. If G is a complete multipartite graph with parts of even sizes at least $m + 2$ and $|E(G)| \equiv 0 \pmod{m}$, then there is a decomposition of G into m-cycles.

The End