Packing of isomorphic induced independent subgraphs

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Notation

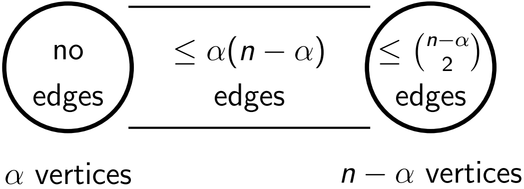
- G - a finite, undirected, and simple graph
- vertex set $V = \{1, ..., n\}$
- m edges
- minimum degree δ (assume $\delta \geq 1$)
- maximum degree Δ
- independence number α

J. Håstad, Clique is Hard to Approximate within $n^{1-\epsilon}$, Acta Mathematica, 182(1999)105-142.

- Independence is hard
- Lower bounds on the independence number α are popular topics in research , however,
- upper bounds have barely been touched in the literature.
- We give an overview to upper bounds,
- we present new ones, and
- we generalize the independence concept to packing of subgraphs into a graph.

A trivial upper bound on α

The number m of edges of G is at most $\binom{n-\alpha}{2} + \alpha(n-\alpha)$.



$$\alpha \le \frac{1+\sqrt{4(n^2-n-2m)+1}}{2}.$$

J.H., D. Rautenbach, 2011

$$\alpha \ge \frac{2n^2}{2m+n+2+\sqrt{(2m+n+2)^2-8n^2}}$$

Further upper bounds on α

Let $\lambda_1 \le ... \le \lambda_n$ be the eigenvalues of G (i.e. the eigenvalues of the adjacency matrix A of G).

Cauchy's inequalities, Interlacing theorem

If U is an induced subgraph of G with eigenvalues $\phi_1 \le ... \le \phi_t$, then $\lambda_i \le \phi_i \le \lambda_{n-t+i}$ for $i = 1, ..., t$.

Let U be a maximum independent set , then $t = \alpha$ and $\phi_1 = ... = \phi_\alpha = 0$.

D. M. Cvetković, 1971

$\alpha \le \min\{|\{i \in \{1, ..., n\} | \lambda_i \le 0\}|, |\{i \in \{1, ..., n\} | \lambda_i \ge 0\}|\}$

Further upper bounds on α

If G is r-regular, then $-r \leq \lambda_1 \leq ... \leq \lambda_n = r$.

A. J. Hoffman, unpublished

If G is r-regular, then

$$\alpha \leq \frac{-\lambda_1}{r - \lambda_1} n.$$

W. H. Haemers, 1979

If δ is the minimum degree of G, then

$$\alpha \leq \frac{-\lambda_1 \lambda_n}{\delta^2 - \lambda_1 \lambda_n} n.$$

To our knowledge, no further non-trivial upper bounds on α are known.

Example: Petersen Graph

- Eigenvalues: $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$
- $\alpha = 4$
- $\left\lfloor \frac{1+\sqrt{4(n^2-n-2m)+1}}{2} \right\rfloor = 8$
- Haemers, Hofmann:
  $\frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n} n = \frac{-\lambda_1}{r-\lambda_1} n = \frac{2}{5} \cdot 10 = 4$
- Cvetković:
  $\alpha \leq \min\{|\{i|\lambda_i \leq 0\}| \; , \; |\{i|\lambda_i \geq 0\}|\} = 4$
  $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$

H-packing of G

- Given a large graph G and a small graph H.
- H-packing: copies $H_1, ..., H_p$ of H as pairwise vertex disjoint subgraphs of G
- strict H-packing: copies $H_1, ..., H_p$ of H as pairwise vertex disjoint and induced subgraphs of G
- maximize p !
- H-factor: the union of $H_1, ..., H_p$ covers all vertices of G
- the H-packing problem (decide whether G admits an H-factor) is NP-complete for both packing versions, if H has a component on at least three vertices , otherwise easy (P. Hell, D.G. Kirkpatrick, 1978, 1983).
- independent induced H-packing: copies $H_1, ..., H_p$ of H as pairwise independent and induced subgraphs of G

Why independent induced H-packing ?

- independent induced K_1-packing is the ordinary independence problem (already hard in this case).
- For our intention to derive upper bounds on the number p of packed copies $H_1, \dots, H_p$ of H in terms of eigenvalues (or generalized eigenvalues) of G and of H, it is important that $H_1, \dots, H_p$ are both pairwise independent and induced subgraphs of G.
- $\alpha(G, H)$ denotes the maximum number of copies of H contained in G as pairwise independent and induced subgraphs.
- $\alpha(G, K_1) = \alpha$

Relationships between H-packing concepts

- $S(G)$ - subdivision graph of G , $L(G)$ - line graph of G
- If $\{H_1, ..., H_p\}$ is an H-packing or a strict H-packing of G and $p \leq ???$, then
- $\{S(H_1), ..., S(H_p)\}$ is an independent induced $S(H)$-packing of $S(G)$, hence, $p \leq \alpha(S(G), S(H))$, and
- $\{L(H_1), ..., L(H_p)\}$ is an independent induced $L(H)$-packing of $L(G)$, hence, $p \leq \alpha(L(G), L(H))$.
- There are relationships between the sets of eigenvalues of G, $S(G)$, and $L(G)$ (D. M. Cvetković, M. Doob, H. Sachs, 1995).

D-eigenvalues of G

- A - the adjacent matrix of G,
  $det(A - \lambda E) = 0$
- D - the degree matrix of G
  (positive definite, because $\delta \geq 1$),
  $det(A - \mu D) = 0$
- Let $-1 \leq \mu_1 \leq ... \leq \mu_n = 1$ be the D-eigenvalues (normalized eigenvalues) of G.
- $-1 = \mu_1$ if and only if G is bipartite

J.H., S. Richter, H. Sachs, Packing of isomorphic induced independent subgraphs, 2013

J.H., S. Richter, H. Sachs, Packing of isomorphic induced independent subgraphs, 2013

Horst Sachs

J.H., S. Richter, H. Sachs, Packing of isomorphic induced independent subgraphs, 2013

Notation: $|V(H)| = h$, $|E(H)| = e$, $\eta_1 \leq \ldots \leq \eta_h$ the eigenvalues of H.

$\alpha(G, H) \leq \frac{2e+h+\sqrt{4h^2(n^2-n-2m)+(2e+h)^2}}{2h^2}$

(generalizing the trivial upper bound $\frac{1+\sqrt{4(n^2-n-2m)+1}}{2}$ on α)

$\alpha(G, H) \leq \frac{\alpha(G,K_1)}{\alpha(H,K_1)}$

$\alpha(G, H) \leq \frac{4e-2\mu_1 h\delta}{(1-\mu_1)\delta^2 h^2} m$,

$\alpha(G, H) \leq \sqrt[3]{\frac{(\frac{n^2}{2m}+(\frac{1}{\delta}-\frac{1}{\Delta})(n-1-\frac{2m}{n}))^2(4me-2\mu_1 m\Delta h)}{(1-\mu_1)h^4}}$, *and*

$\alpha(G, H) \leq \min\{\frac{1}{q}|\{i \mid \lambda_i \leq \eta_q\}|, \frac{1}{h-q+1}|\{i \mid \lambda_i \geq \eta_q\}|\}$ *for all* $q \in \{1, \ldots, h\}$.

To our knowledge, no further non-trivial upper bound is known in case $H \neq K_1$.

Example: G - Petersen Graph, $H = P_3$

- $\alpha(G, P_3) = 1$, $n = 10$, $m = 15$, $\mu_1 = -\frac{2}{3}$, $h = 3$, $e = 2$
- $\left\lfloor \frac{2e+h+\sqrt{4h^2(n^2-n-2m)+(2e+h)^2}}{2h^2} \right\rfloor = 3$
- $\left\lfloor \frac{4e-2\mu_1 h\delta}{(1-\mu_1)\delta^2 h^2} m \right\rfloor = 3$
- $\left\lfloor \sqrt[3]{\frac{(\frac{n^2}{2m}+(\frac{1}{\delta}-\frac{1}{\Delta})(n-1-\frac{2m}{n}))^2(4me-2\mu_1 m\Delta h)}{(1-\mu_1)h^4}} \right\rfloor = 7$
- eigenvalues λ of G: $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$
- eigenvalues η of P_3: $-\sqrt{2}, 0, \sqrt{2}$

  ● ● ● ● ● ● ● ● ● ● ● ● ● (interlacing)
- $\min\{\frac{1}{q}|\{i \mid \lambda_i \leq \eta_q\}|, \frac{1}{4-q}|\{i \mid \lambda_i \geq \eta_q\}|\}$ for all $q \in \{1, 2, 3\}$
- left: $\frac{4}{1}$, $\frac{4}{2}$, $\frac{9}{3}$ right: $\frac{6}{3}$, $\frac{6}{2}$, $\frac{1}{1} = 1$

Again α $(H = K_1,\ h = 1, \text{ and } e = 0)$

$$\alpha \le \frac{-2\mu_1}{(1-\mu_1)} \cdot \frac{m}{\delta}$$

$$\alpha \le \sqrt[3]{\frac{(\frac{n^2}{2m} + (\frac{1}{\delta} - \frac{1}{\Delta})(n-1-\frac{2m}{n}))^2(-2\mu_1 \Delta m)}{(1-\mu_1)}}$$

- $\alpha \le \frac{1+\sqrt{4(n^2-n-2m)+1}}{2}$ (trivial) and $\alpha \le \frac{-\lambda_1\lambda_n}{\delta^2-\lambda_1\lambda_n} n$ (Haemers).
- These four bounds are pairwise not comparable.
- The fifth upper bound is Cvetković's bound $\min\{|\{i \in \{1,\dots,n\} | \lambda_i \le 0\}|, |\{i \in \{1,\dots,n\} | \lambda_i \ge 0\}|\}$.
- To our knowledge, no further non-trivial upper bound on α is known.

Thank you for your attention !