Cyclic decompositions of complete and complete multipartite hypergraphs

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Outline

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Cyclic partitions

Definition

A **cyclic t-partition** of a hypergraph (V, E) is a partition of the (hyper)edge set E of the form $\{F, F^{\theta}, F^{\theta^2}, \dots, F^{\theta^{t-1}}\}$, where θ is a permutation of the vertex set V.

A cyclic t-partition of a hypergraph (V, E) is a decomposition of the hypergraph into t isomorphic hypergraphs which are permuted cyclically by a permutation θ of V.

Complete uniform hypergraphs

Definition

The **complete k-uniform hypergraph** with vertex set V has edge set $\binom{V}{k}$, the set of all k-element subsets of V.

We assume that a hypergraph with n vertices has vertex set $V_n = \{1, 2, \ldots, n\}$.

The complete k-uniform hypergraph on n vertices is denoted by $\mathcal{K}_n^{(k)}$.

t-complementary hypergraphs

- A cyclic 2-partition of the complete graph $\mathcal{K}_n^{(2)}$ contains a self-complementary graph and its complement.
- An isomorphism between a self-complementary graph and its complement is called a **complementing permutation** .
- Analogously, each of the t k-uniform hypergraphs in a cyclic t-partition of $K_n^{(k)}$ is called **t-complementary** , and the associated permutation θ is called a **$(\mathbf{t}, \mathbf{k})$-complementing permutation** .

- The problem of determining whether a given uniform hypergraph is t-complementary has the same complexity as the graph isomorphism problem. (M.J. Colbourn and C.J. Colbourn, 1978)
- Instead, we solve the problem of determining whether a given permutation is a (t, k)-complementing permutation for some t-complementary k-uniform hypergraph.
- We characterize the cycle type of such permutations.

Construct a 2-complementary 3-hypergraph of order 6.

$\theta = (1\ 2)(3\ 4\ 5\ 6)$

Let $A_1 = \{x, y, z\} \subset \{1, 2, 3, 4, 5, 6\}$

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Take either the red or the blue edges from each orbit of θ on the 3-subsets of $V = \{1,2,3,4,5,6\}$. There are 2^6 self-complementary 3-hypergraphs on V with $(2,3)$-complementing permutation θ (at most 2^5 up to isomorphism).

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Construct a 2-complementary 4-hypergraph of order 18.

$\theta = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18)$

$A_1 = \{1, 4, 7, 10\}$ $\{2, 5, 8, 11\}$ $\{3, 6, 9, 12\}$

......

An orbit of θ on the 4-subsets of $\mathbb{Z}_{18}$ has odd length. This implies that θ is not a (2,4)-complementing permutation of V_{18}.

Construct a 4-complementary graph on $\mathbb{Z}_8$

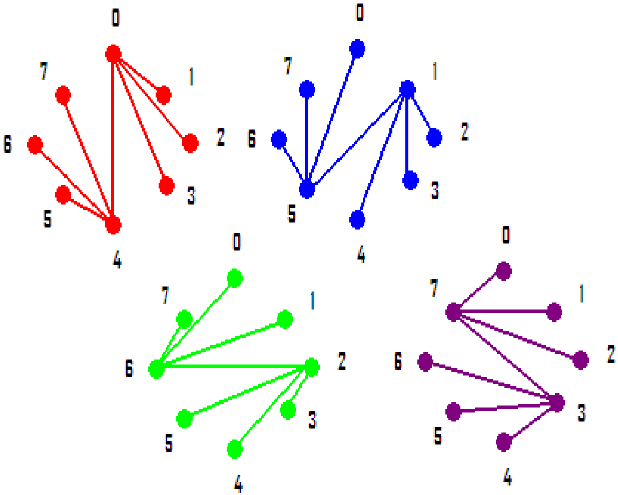
$\theta = (0, 1, 2, 3, 4, 5, 6, 7)$
Color the orbits of θ on the 2-subsets of $\mathbb{Z}_8$:

$\mathcal{O}_1$: 01 12 23 34 45 56 67 07
$\mathcal{O}_2$: 02 13 24 35 46 57 06 17
$\mathcal{O}_3$: 03 14 25 36 47 05 16 27
$\mathcal{O}_4$: 04 15 26 37

Every orbit has length divisible by 4. Choose the edges of one color from each orbit.

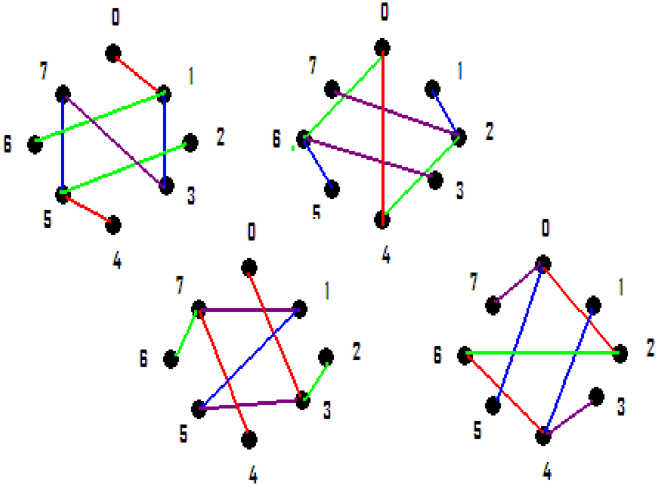
There are 4^4 different 4-complementary graphs on $\mathbb{Z}_8$ with $(4, 2)$-complementing permutation θ (at most 4^3 up to isomorphism).

Eg. Choose the same color from each orbit ...



Eg. Choose a different color from every orbit ...

$\mathcal{O}_1$: red, $\mathcal{O}_2$: blue, $\mathcal{O}_3$: green, $\mathcal{O}_4$: violet.



A permutation θ of V is a (t, k)-complementing permutation

$\iff$ the sequence $A, A^{\theta}, A^{\theta^2}, A^{\theta^3}, \ldots$ has length divisible by t ,
for all k-subsets A of V,

$\iff$ $A^{\theta^j} \neq A$,
for all k-subsets A of V, for all integers $j \not\equiv 0 \pmod{t}$.

(Gosselin, Szymański, Wojda, 2010)

Theorem

Let n, k, p and α be positive integers such that $k < n$ and p is prime. Let θ be a permutation on V_n. Then θ is a (p^α, k)-complementing permutation if and only if there is an integer $\ell \geq 0$ such that the union of the orbits of θ with cardinality not divisible by $p^{\ell+\alpha}$ has cardinality less than $k \bmod p^{\ell+1}$.

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Theorem

Let n, k, p and α be positive integers such that $k < n$ and p is prime. Suppose that $k = \sum_{i\geq 0} k_i p^i$ and $n = \sum_{i\geq 0} n_i p^i$, where $0 \leq k_i < p$ and $0 \leq n_i < p$ for $i \in \mathbb{N}$. Then the following three statements are equivalent.

1. *There exists a cyclic p^α-partition of $\mathcal{K}_n^{(k)}$.*
2. *There exists $\ell \in \mathbb{N}$ such that $k_\ell \neq 0$ and $n \mod p^{\ell+\alpha} < k \mod p^{\ell+1}$.*
3. *There exist $r, \ell \in \mathbb{N}$ with $r \leq \ell$ such that $n_r < k_r$, $n_i = 0$ for $\ell < i < \ell + \alpha$ whenever $\alpha > 1$, and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$.*

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Condition **3** means that the base-p representations of n and k have the following form:

	$\dots$	$\ell+\alpha$	$\ell+\alpha-1$	$\dots$	$\ell+1$	ℓ	$\dots$	$r+1$	r	$r-1$	$\dots$
n	$\dots$	$*$	0	$\dots$	0	n_ℓ	$\dots$	n_{r+1}	n_r	$*$	$\dots$
						$\shortparallel$	$\dots$	$\shortparallel$	$\wedge$		
k	$\dots$	$*$	$*$	$\dots$	$*$	k_ℓ	$\dots$	k_{r+1}	k_r	$*$	$\dots$

Lemma

If $q_1 q_2 \cdots q_m$ is the prime power factorization of t, then θ is a (t,k)-complementing permutation if and only if θ is a (q_i,k)-complementing permutation for all $i \in \{1,2,\ldots,m\}$.

Corollary

Let t, k and n be positive integers, $k \leq n$, let $q_1 q_2 \cdots q_m$ be the prime power factorization of t, where $q_i = p_i^{\alpha_i}$ for $1 \leq i \leq m$. If there exists a cyclic t-partition of $K_n^{(k)}$, then for each $i \in \{1,2,\ldots,m\}$ there is $\ell_i \in \mathbb{N}$ such that $k_{\ell_i} \neq 0$ and

$$\mathbf{n} \bmod \mathbf{p}_\mathbf{i}^{\ell_\mathbf{i}+\alpha_\mathbf{i}} < \mathbf{k} \bmod \mathbf{p}^{\ell_\mathbf{i}+1}$$

.

Every permutation of V_{89} with one orbit of cardinality 64 and one orbit of cardinality 25 is $(2, 40)$-complementing.
$(2 = 2^1$, $40 = 2^5 + 2^3$, and
$89 \bmod 2^{5+1} = 25 < 40 = 40 \bmod 2^{5+1}$.)

Every permutation of V_{89} with one orbit of cardinality 8 and one orbit of cardinality 81 is $(9, 40)$-complementing.
$(9 = 3^2$, $40 = 3^3 + 3^2 + 3^1 + 3^0$, and
$89 \bmod 3^{2+2} = 8 < 13 = 40 \bmod 3^{2+1}$.)

However, there is no $(18, 40)$-complementing permutation of V_{89}.

Hence the necessary condition of the previous corollary is not sufficient.

Complete nonuniform hypergraphs

Definition

For a nonempty subset K of V_{n-1}, the **complete K-hypergraph of order n** is $\left(V_n, \bigcup_{k \in K} \binom{V_n}{k}\right)$ and denoted by $\mathcal{K}_n^{(K)}$.

If $\{E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{t-1}}\}$ is a cyclic t-partition of $\mathcal{K}_n^{(K)}$, then θ is called a **(t, K)-complementing permutation**.

(Gosselin, Szymański, Wojda, 2010)

Lemma

Let $K \subseteq V_{n-1}$. A permutation θ of V_n is a (t, K)-complementing permutation if and only if θ is a (t, k)-complementing permutation for all $k \in K$.

Theorem

Let n, k, p and α be positive integers such that p is prime and $k < n$. A permutation θ of V_n is (p^α, V_k)-complementing if and only if the cardinality of any orbit of θ is divisible by $p^{\alpha+\beta}$, where $\beta = \lfloor \log_p k \rfloor$.

Corollary

Let n, k, p and α be positive integers such that p is prime and $k < n$. There is a cyclic p^α-partition of $\mathcal{K}_n^{(V_k)}$ if and only if $p^{\alpha+\beta}$ divides n, where $\beta = \lfloor \log_p k \rfloor$.

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(Gosselin, Szymański, Wojda, 2010)

Theorem

Let n, k and q be positive integers such that $t = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$, where $p_1, p_2, \ldots, p_s$ are mutually distinct primes, $k < n$, and $\beta_j = \lfloor \log_{p_j} k \rfloor$ for every $j = 1, 2, .., s$. A permutation θ of V_n is (t, V_k)-complementing if and only if the cardinality of any orbit of θ is divisible by $p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdot \ldots \cdot p_s^{\alpha_s+\beta_s}$.

Outline

Complete multipartite hypergraphs

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Cyclic partitions of complete multipartite hypergraphs

Let k, p, t and α be positive integers such that p is prime. Let $V = A_1 \cup A_2 \cup ... \cup A_t$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

Let θ be a permutation of the set V such that ${A_i}^{\theta} = A_i$ for every $i = 1, 2, ..., t$.

When does θ induce a cyclic p^{α}-partition of $\mathcal{K}^{(V_k)}(A_1, A_2, ..., A_t)$?

When the orbit of θ on any $e \in E_r$ has length divisible by p^{α}, for all $r \in V_k$.

This is guaranteed if $\theta_{|A_i}$ is a (p^{α}, V_{k-1})-complementing permutation for all but at most one i,

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Theorem

... Each of the following two conditions is sufficient for θ to induce a cyclic p^α-partition of $\mathcal{K}^{(V_k)}(A_1, A_2, ..., A_t)$:

1. *For all but at most one $i \in \{1, 2, ..., t\}$, the cardinality of all the orbits of $\theta_{|A_i}$ are divisible by $p^{\alpha+\beta}$, where $\beta = \lfloor \log_p(k-1) \rfloor$.*
2. *For every $i \in \{1, 2, ..., t\}$, the cardinalities of all orbits of $\theta_{|A_i}$ are divisible by $p^{\alpha+\gamma}$, where $\gamma = \lfloor \log_p k/2 \rfloor$.*

Corollary

Let $n_1, n_2, \ldots, n_t, k, p$ and α be positive integers such that p is prime. If at least one of the following two conditions is verified then there is a cyclic p^{α}-partition of $\mathcal{K}^{(V_k)}_{n_1,n_2,\ldots,n_t}$.

1. *For all but at most one $i \in \{1, 2, \ldots, t\}$, $p^{\alpha+\beta} | n_i$ where $\beta = \lfloor \log_p(k-1) \rfloor$.*
2. *For every $i \in \{1, 2, \ldots, t\}$, $p^{\alpha+\gamma} | n_i$ where $\gamma = \lfloor \log_p k/2 \rfloor$.*

Thank You!