Algebraic Hypergraph Decompositions

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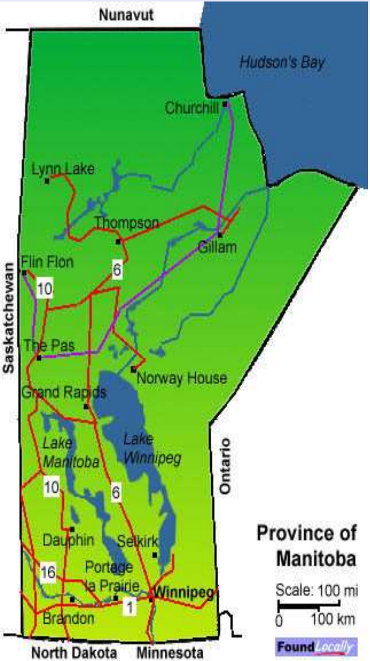
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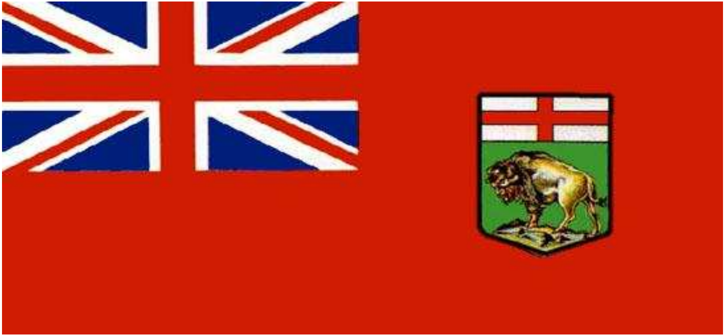

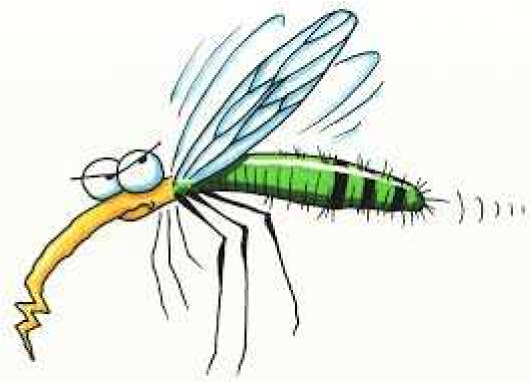

Outline

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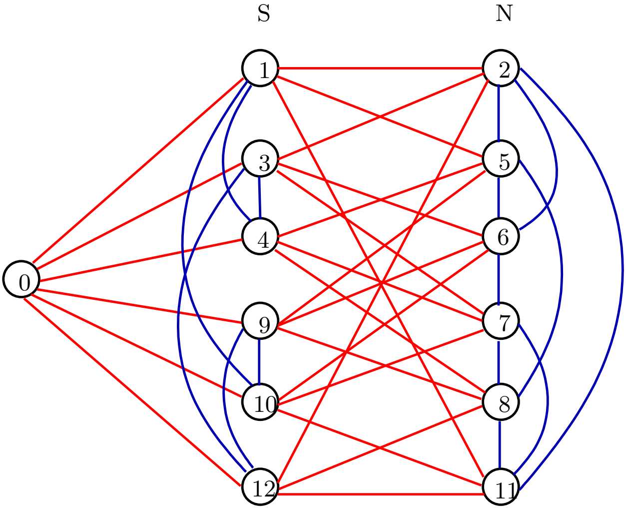
The Paley graph

Definition

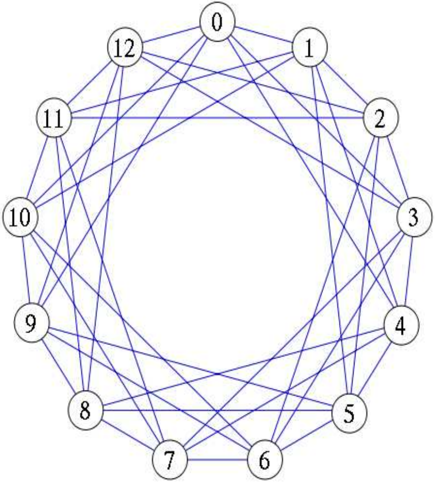
For a prime power $q \equiv 1 \pmod 4$ and a finite field $\mathbb{F}_q$, the **Paley graph of order q**, denoted by **Paley(q)**, is the simple graph with vertex set $V = \mathbb{F}_q$ and edge set E, where

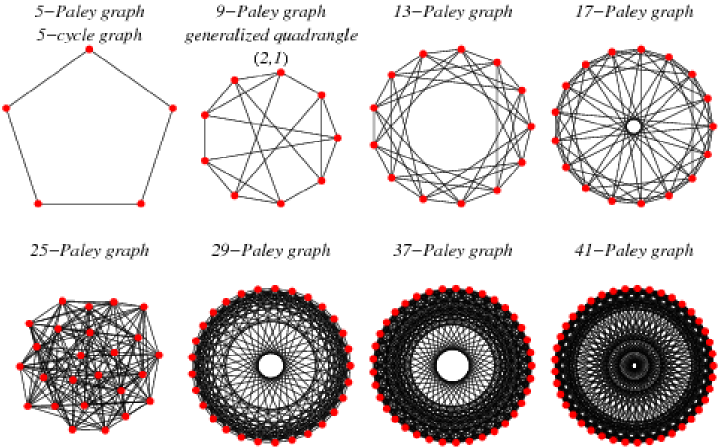
$$\{x, y\} \in E \iff x - y \text{ is a nonzero square.}$$

Paley(13)



$Paley(13)$




5−Paley graph
5−cycle graph
9−Paley graph
generalized quadrangle
(2,1)
13−Paley graph
17−Paley graph
25−Paley graph
29−Paley graph
37−Paley graph
41−Paley graph

Paley(q) is self-complementary

If ω is a generator of $\mathbb{F}_q^*$, then

$$x - y \in \langle \omega^2 \rangle \iff \omega x - \omega y = \omega(x - y) \notin \langle \omega^2 \rangle.$$

$\mathbf{T}_{\omega,\mathbf{0}} : x \mapsto \omega x$ is an isomorphism from *Paley*(q) to its complement. $\square$

$Paley(q)$ is vertex-transitive

For $b \in \mathbb{F}_q$,

$$x - y \in \langle \omega^2 \rangle \iff (x+b) - (y+b) = x - y \in \langle \omega^2 \rangle$$

$\mathbf{T_{1,b}} : x \mapsto x + b$ is an automorphism of $Paley(q)$.

$\{T_{1,b} : b \in \mathbb{F}_q\}$ acts transitively on $\mathbb{F}_q$. $\square$

$Aut(Paley(q))$ is an index-2 subgroup of the affine group $A\Gamma L(1, q)$

Outline

Paley graphs

t-complementary hypergraphs

Generalized Paley hypergraphs

Definition

A simple k-uniform hypergraph X with vertex set V and edge set E is **t-complementary** if there is a permutation θ on V such that the sets

$$E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$$

partition the set of k-subsets of V.

θ is called a **t-antimorphism** of X (i.e., $\theta \in \mathbf{Ant_t(X)}$).

- The 2-complementary 2-uniform hypergraphs are the **self-complementary graphs**, which have been well studied due to their connection to the graph isomorphism problem.
- The t-complementary k-hypergraphs correspond to **cyclic edge decompositions (cyclotomic factorisations)** of the complete k-uniform hypergraph into t parts.
- The vertex-transitive t-complementary k-uniform hypergraphs correspond to **large sets of isomorphic designs** which are point-transitive.

Outline

Paley graphs

t-complementary hypergraphs

Generalized Paley hypergraphs

The Paley graph - revisited

Definition

For a prime power $q \equiv 1 \pmod 4$ and a finite field $\mathbb{F}_q$ of order q, the **Paley graph of order q**, denoted by $\mathbf{Paley(q) = (V, E)}$, is the simple graph with $\mathbf{V = \mathbb{F}_q}$ and

$$\{\mathbf{x}, \mathbf{y}\} \in \mathbf{E} \iff \mathbf{x} - \mathbf{y} \in \langle \omega^{\mathbf{2}} \rangle$$

where ω is a generator of $\mathbb{F}_q^*$.

Constructing t-complementary k-hypergraphs

Partition a group G into t sets

$$\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{t-1},$$

where each $\mathcal{C}_i$ is a union of cosets of a subgroup S of G.

Find an operation $\Psi : \binom{V}{k} \to G$ and a permutation $\theta : V \to V$ such that

$$\Psi(\{x_1, \ldots, x_k\}) \in \mathcal{C}_i \iff \Psi(\{x_1, \ldots, x_k\}^\theta) \in \mathcal{C}_{i+s}$$

for some s where $\gcd(s, t) = 1$.

Let $E_i = \left\{ e \in \binom{V}{k} \ : \ \Psi(e) \in \mathcal{C}_i \right\}$.

Then $X_i = (V, E_i)$ is t-complementary with t-antimorphism θ.

Examples

1. Paley Graphs:
   - $V = \mathbb{F}_q$.
   - $G = \mathbb{F}_q^*$.
   - $S = \langle \omega^2 \rangle$.
   - $\Psi(\{x, y\}) = x - y$.
2. Generalized Paley k-hypergraphs :
   - $V = \mathbb{F}_q$.
   - G is the group of squares of $\mathbb{F}_q^*$.
   - $S = \langle \omega^{2t\binom{k}{2}} \rangle$
   - $\Psi(\{x_1, x_2, \ldots, x_k\}) = \prod_{i<j}(x_i - x_j)^2$.

The Generalized Paley Hypergraph $Paley(q, k, t)$

Definition

t is prime, ℓ is the highest power of t dividing k or $k-1$.
q is a prime power, $q \equiv 1 \pmod{t^{\ell+1}}$
G is the group of squares in $\mathbb{F}_q^*$.
$S = \langle \omega^{2t\binom{k}{2}} \rangle$.
$c = \gcd(|G|, \binom{k}{2})$. ($tc$ is the number of cosets of S in G.)
F_i is the coset $\omega^{2i}\langle \omega^{2t\binom{k}{2}} \rangle$ in G $(0 \leq i \leq tc-1)$.
$\mathcal{C}_j = F_{jc+0} \cup F_{jc+1} \cup \cdots \cup F_{(j+1)c-1}$ $(0 \leq j \leq t-1)$.

The **Generalized Paley Hypergraph** $\mathbf{Paley(q, k, t)} = (V, E)$ is the simple k-hypergraph with $\mathbf{V} = \mathbb{F}_\mathbf{q}$ and

$$\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\} \in \mathbf{E} \iff \prod_{\mathbf{i<j}} (\mathbf{x_i} - \mathbf{x_j})^2 \in \mathcal{C}_\mathbf{0}.$$

$P(q, k, t)$ is t-complementary

$$\prod_{i<j}(x_i - x_j)^2 \in F_i$$
$$\iff \prod_{i<j}(\omega x_i - \omega x_j)^2 = \omega^{2\binom{k}{2}} \prod_{i<j}(x_i - x_j)^2 \in F_{i+sc},$$

where $\gcd(s, t) = 1$.

$\mathbf{T}_{\omega,\mathbf{0}} : \mathbf{x} \to \omega\mathbf{x}$ is a t-antimorphism of $Paley(q, k, t)$.

$Paley(q, k, t)$ is vertex-transitive

For $b \in \mathbb{F}_q$,

$$\prod_{i<j}(x_i - x_j)^2 \in F_i$$
$$\iff \prod_{i<j}((x_i + b) - (x_j + b))^2 = \prod_{i<j}(x_i - x_j)^2 \in F_i.$$

$\mathbf{T_{1,b}} : \mathbf{x} \to \mathbf{x} + \mathbf{b}$ is an automorphism of $Paley(q, k, t)$.

Automorphisms and t-antimorphisms of $Paley(q, k, t)$

$$Aut(Paley(q, k, t)) \geq \{T_{a,b} \mid a = \omega^s, s \equiv 0 \pmod{t}, b \in \mathbb{F}_q\}$$

$$Ant_t(Paley(q, k, t)) \supseteq \{T_{a,b} \mid a = \omega^s, s \not\equiv 0 \pmod{t}, b \in \mathbb{F}_q\}.$$

$\mathbf{T_{a,b} : x \mapsto ax + b}$

$Aut(Paley(q, k, t))$ contains an index-t subgroup of the affine group $A\Gamma L(1, q)$.

Generalized Paley hypergraph constructions

$t = 2$, $k = 2$ (Paley)
$t = 2$, $k = 3$ (Kocay, 1992)
$t = 2$, $k = 2$, r-factor (Peisert, 2001)
t, $k = 2$ (Li, Praeger 2003)(Li, Lim and Praeger 2009)
$t = 2$, any k (Potočnik and Šajna, 2009)
Odd prime t, any k, r-factor (G. 2011)

n not a prime power?

Construction: Generalized Paley k-hypergraph

- $n \geq k$, $n = q_1 q_2 \cdots q_s$ is the prime power decomposition of n.
- ℓ is the largest power of t that divides m or $m-1$ for $2 \leq m \leq k$.
- $q_i \equiv 1 \pmod{t^{\ell+1}}$ for $i = 1, 2, \ldots, s$.
- $\mathcal{V} := \mathbb{F}_{q_1} \times \mathbb{F}_{q_2} \times \cdots \times \mathbb{F}_{q_s}$.
- Define $\mathbf{P}aley(n, k, t) = (\mathcal{V}, \mathcal{E})$, where ....

$$E = \{\{x_{11}, x_{12}, \ldots, x_{1j}, \ldots, x_{1s}\}$$
$$\{x_{21}, x_{22}, \ldots, x_{2j}, \ldots, x_{2s}\}$$
$$\{x_{31}, x_{32}, \ldots, x_{3j}, \ldots, x_{3s}\}$$
$$\bullet$$
$$\bullet$$
$$\bullet$$
$$\{x_{k1}, x_{k2}, \ldots, x_{kj}, \ldots, x_{ks}\}\}$$

j is the smallest integer in $\{1, 2, \ldots, s\}$ for which j-th coordinates of the elements in E are not all equal.

$E \in \mathcal{E}$ if and only if the j-th coordinates form an edge of $\mathbf{Paley(q_j, k, t)}$.

$\mathbf{Paley(n, k, t)}$ is vertex-transitive and t-complementary.

Conditions on order

Theorem

Let t be prime, let ℓ and b be positive integers such that $1 \leq b \leq t-1$, and suppose that k or $k-1$ equals bt^{ℓ}. Suppose n is a positive integer, $n > k$, and $n = q_1 q_2 \cdots q_s$ is its prime power decomposition. If $n \equiv 1 \pmod{t^{\ell+1}}$, then there exists a vertex-transitive t-complementary k-hypergraph of order n if and only if

$$q_i \equiv 1 \pmod{t^{\ell+1}} \quad \text{for } 1 \leq i \leq s.$$

Necessity: $t = 2$, $k = 2$ (Muzychuk, 1992);
$t = 2$ (Potočnik, Šajna, 2007); t prime (G. 2010)
Sufficiency: $t = 2$, $k = 2$ (Rao, 1985);
$t = 2$, any k, n a prime power (Potočnik, Šajna, 2009);
t prime, any k, any n (G. 2010)

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Raymond Paley (1907-1933)