Automated Conjecture Making for the Independence Number Project

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An independent set is a set of vertices of a graph, of which no two are pairwise adjacent. The independence number $\alpha(G)$ of a graph G is the size of the largest independent set in G.

Independence number is hard to compute. All known general algorithms to find it are exponential.

However, independence number is easy to compute for particular classes of graphs such as, for example, bipartite graphs or claw-free graphs.

The goal of the Independence Number Project (INP) is to expand the class of graphs for which the independence number is easy to compute.

Let $l(G)$ be the minimum of all known efficiently computable lower bounds for $\alpha(G)$.

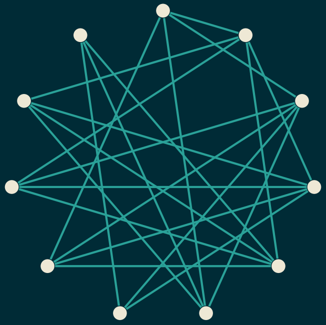
Let $u(G)$ be the maximum of all known efficiently computable upper bounds for $\alpha(G)$.

Independence number can be computed efficiently for all graphs where $l(G) = u(G)$.

One way to advance our goal is to find new upper and lower bounds for α.

To do this, we search for the smallest graph that does not belong to one of the easy-to-compute classes where $l(G) \neq u(G)$. We call this a difficult graph.

Here is our current difficult graph:



graph6: J?`FBo{fdb?

For this particular difficult graph, we already have an upper bound that equals α: the Lovász theta function. We just need a lower bound that also equals α!

Since this difficult graph may represent a gap in theory, we wish to find new conjectures for bounds on α in order to inspire research.

We can automate the process of creating new conjectures using a heuristic created by Fajtlowicz.

This technique has been used in software such as Graffiti.pc, and conjectures created by software have led to some nice results.

Theorem (Chung, 1988)

The average distance between distinct vertices of a graph is less than or equal to the independence number.

The algorithm requires these inputs:

- Arithmetic operations, e.g. $+, -, \times$
- Graph invariants, e.g. diameter, radius
- Graphs $G_1, \ldots, G_n$

First, we assemble all permutations of the arithmetic operations on the graph invariants, creating expressions to be tested.

- diameter + diameter
- diameter × diameter
- diameter + radius
- diameter − radius
- diameter × radius
- *etc*...

Next, each of these expressions is evaluated for each of the input graphs, and tested against α. For example, if the 5-cycle is one of the input graphs,

$$\text{diameter} + \text{radius} = 2 + 2 = 4 \not\leq \alpha = 2$$

So this expression is discarded. Only expressions which are true for all input graphs are kept.

We also check each expression for significance, that is, if there is an input graph G such that the expression $x(G)$ is larger than $x'(G)$ for all other expressions x'.

We stop searching when we find an expression that gives equality to α for each graph, or when we reach our complexity limit.

We begin our search by inputting just our current difficult graph.

Graphs

- J?`FBo{fdb?

Conjectures

- $2 * \text{diameter} \leq \alpha$

Graphs

- J?`FBo{fdb?
- P_2

Conjectures

- $\text{diameter}^2 \leq \alpha$
- $\text{avg. distance}^2 \leq \alpha$

Graphs

- J?`FBo{fdb?
- P_2
- P_3

Conjectures

- annihilation $-$ diameter $\leq \alpha$
- avg. distance^2 $\leq \alpha$
- diameter $\leq \alpha$

Graphs

- J?`FBo{fdb?
- P_2
- P_3
- P_4

Conjectures

- annihilation $-$ diameter $\leq \alpha$
- avg. distance^2 $\leq \alpha$
- radius $\leq \alpha$
- residue $\leq \alpha$

Graphs

- J?`FBo{fdb?
- P_2
- P_3
- P_4
- K_6

Conjectures

- radius $*$ residue $-$ dia. $\leq \alpha$
- avg. distance $\leq \alpha$
- radius $\leq \alpha$
- residue $\leq \alpha$

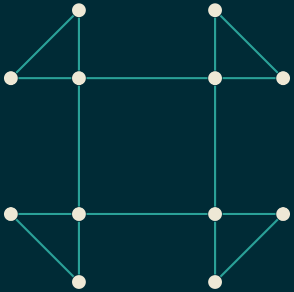
Graphs

- J?`FBo{fdb?
- P_2
- P_3
- P_4
- K_6
- KIaIAC_GG_A@

Conjectures

- $2 * \text{residue} - \text{diameter} \leq \alpha$
- avg. distance $\leq \alpha$
- radius $\leq \alpha$
- residue $\leq \alpha$

Counterexample to radius $*$ residue $-$ diameter $\leq \alpha$:



graph6: KlaIAC_GG_A@

Although these conjectures may be false for all graphs, they may hold true for particular classes of graphs, such as claw-free or triangle-free.

All graphs of order 10 or less are not difficult, that is, they either belong to a class where computing α is efficient, or there exist upper and lower bounds on α with equality.

Our software is written in Python using Sage, and is open source.

http://github.com/IndependenceNumberProject/inp

Conjectures

Let G be a graph having diameter D, minimum degree δ, maximum degree Δ, and Szeged index Sz. Then,

$$\frac{\sqrt{\frac{\text{Sz}}{D}}}{\delta} \leq \alpha$$

Also,

$$\frac{\sqrt{\text{Sz}}}{\Delta} \leq \alpha$$

Find a counterexample!