Change of limiting distributions of the number of large matchings

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joint work with Cristiane Sato

Distribution of small subgraphs

Large subgraphs

Matchings

Future work

Probability spaces in this talk

- $\mathcal{G}(n, p)$.
- $\mathcal{G}(n, m)$
- $\mathcal{G}(n, d)$.

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G: a fixed graph. $m(G) = \max\left\{\frac{|E(H)|}{|V(H)|} : H \subseteq G\right\}$.

Let X_n denote the number of subgraphs in $\mathcal{G}(n, p)$ that are isomorphic to G.

- If $np^m = o(1)$, then a.a.s. $X_n = 0$.
- What if $np^m \to c > 0$, for constant c or for c goes to infinity sufficiently slow? (Poisson?)

It is Poisson convergent if and only if G is strictly balanced (in a certain range of p). Bollobás (1981), Karoński & Ruciński (1983), Ruciński & Vince (1986).

- What if $np^m \to \infty$?

Ruciński (1988): if $np^m \to \infty$ and $n^2(1-p) \to \infty$,

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• The number of Hamilton cycles in $\mathcal{G}(n, d)$. Robinson & Wormald 1992, Janson 1995.

The logarithm of this variable can be expressed as a linear combination of an infinite set of independent Poisson variables.

• The number of cycles (size from 3 to n) in $\mathcal{G}(n, d)$, Garmo 1999.

• The number of perfect matchings, Hamilton cycles and spanning trees in $\mathcal{G}(n, p)$ ($p = \omega(n^{-1/2})$) and $\mathcal{G}(n, m)$ ($m = \omega(n^{3/2})$). Janson 1994.

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Main results

Let $X_{n,\ell}$ denote the number of ℓ-matchings (matchings of size ℓ) contained in $\mathcal{G}(n, p)$ as a subgraph.

Theorem

Suppose that $1 - p = \Omega(1)$ and $p \geq n^{-1/8+\epsilon}$. For any integer $\ell = \ell_n \in [1, n/2]$,

(i) *if $\ell = o(n\sqrt{p})$, then $X_{n,\ell}$ is asymptotically normally distributed;*

(ii) *if $\ell = \Omega(n\sqrt{p})$, then $X_{n,\ell}$ is asymptotically log-normally distributed.*

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The subcritical case

- $$\frac{X_{n,\ell} - \mathbf{E}X_{n,\ell}}{\mathbf{Var}X_{n,\ell}} \to \mathcal{N}(0,1).$$

Method of moments for normal distributions

Theorem

If $Z_1, Z_2, \ldots$ are random variables with finite moments and a_n are positive numbers such that, for fixed integer $k \geq 2$, as $n \to \infty$,

$$\mathbf{E}\Big((Z_n - \mathbf{E}Z_n)^k\Big) = \begin{cases} (k-1)!!a_n^k + o(a_n^k), & \text{if } k \text{ is even;} \\ o(a_n^k), & \text{if } k \text{ is odd;} \end{cases}$$

then $(Z_n - \mathbf{E}Z_n)/\mathbf{Var}Z_n \xrightarrow{d} \mathcal{N}(0,1)$.

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Fix an integer $k \geq 2$. Let S be the set of ℓ-matchings of K_n. Then,

$$X_{n,\ell} = \sum_{i \in S} X_i.$$

- $\mathbf{E} X_i = p^\ell$ for every $i \in S$.
- How to compute

$$\begin{aligned} \mathbf{E}\Big((X_{n,\ell} - \mathbf{E}X_{n,\ell})^k\Big) &= \mathbf{E}\left(\left(\sum_{i \in S}(X_i - p^\ell)\right)^k\right) \\ &= \sum_{\mathbf{i} \in S^k} \mathbf{E}(X_{i_1} - p^\ell) \cdots (X_{i_k} - p^\ell) \end{aligned}$$

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Estimate $\sum_{\mathbf{i}\in S^k} \mathbf{E}(X_{i_1} - p^\ell)\cdots(X_{i_k} - p^\ell)$.

- Each summand depends on how $i_1, \ldots, i_k$ intersect with each other;
- We need to find structures of $(i_1, \ldots, i_k)$ that lead the contribution to the summation $\sum_{\mathbf{i}\in S^k}$.

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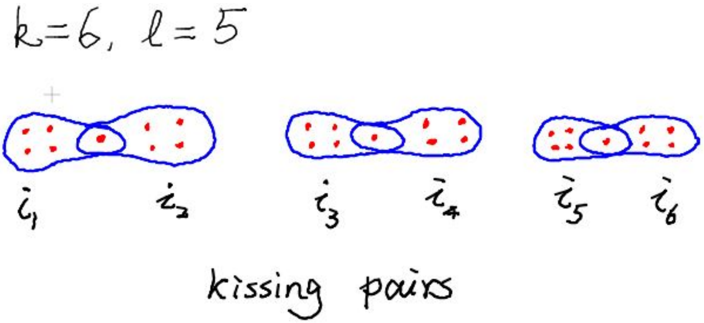
The leading structures.



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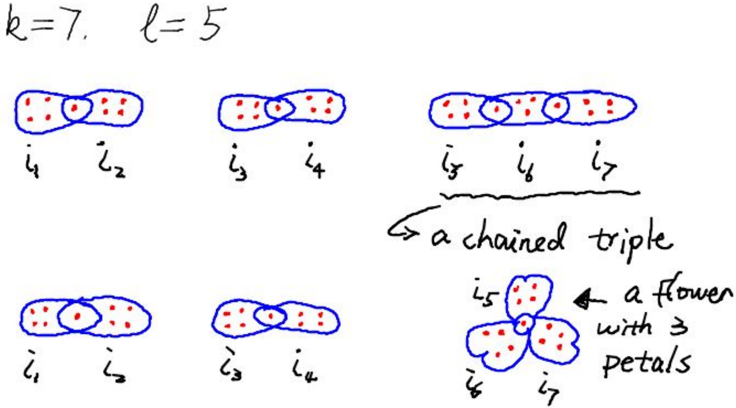

The supercritical case

The method:

- S: a set of graphs, each containing h vertices.
- X_n: the number of graphs in S that are contained in $\mathcal{G}(n,p)$ (or $\mathcal{G}(n,m)$) as a subgraph.
- Let $\mu = \mathbf{E}_{\mathcal{G}(n,m)}(X_n)$ and $\lambda = \mathbf{E}_{\mathcal{G}(n,p)}(X_n)$.
- Let $F(i) = \{(G_1, G_2) \in S^2,\ |E(G_1) \cap E(G_2)| = i\}$; let $f_i = |F(i)|$.

The supercritical case

Theorem (Gao 2012)

Let $r_j = f_j/f_{j-1}$. Assume p is ideal. If the following conditions hold:

(a) $\forall j \leq$ COD, $r_j =$ magical ratio;

(b) r_j WELL BOUNDED for all COD $\leq j \leq$ WHALE;

(c) $t(n) := \sum_{j > WHALE} f_j$ WELL BOUNDED;

Then in $\mathcal{G}(n,p)$, X_n is asymptotically log-normally distributed.

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Let $r_j = f_j/f_{j-1}$. Assume $h^3 = o(p^2 n^4)$ and $h^2 = \Omega(pn^2)$. If for all $m = p\binom{n}{2} + O(\sqrt{pn^2})$, and for $\rho(n) = h^2/m$, there is some function $\gamma(n)$ for which the following conditions hold:

(a) $\forall K > 0,\ 1 \le j \le K\rho(n), r_j = \frac{h^2}{Nj}(1 + o(m/h^2))$;

(b) $r_j \le m/2\binom{n}{2}$ for all $4\rho(n) \le j \le \gamma(n)$;

(c) $t(n) := \sum_{j>\gamma(n)} f_j = o(\mu|S|)$;

Then in $\mathcal{G}(n,p)$, X_n is asymptotically log-normally distributed.

$$\frac{\ln(e^{\beta_n^2/2} X_n/\lambda_n)}{\beta_n} \xrightarrow{d} \mathcal{N}(0,1), \quad as\ n \to \infty,$$

provided that $\liminf_{n\to\infty} \beta_n > 0$, *where* $\beta_n = h\sqrt{(1-p)/p\binom{n}{2}}$.

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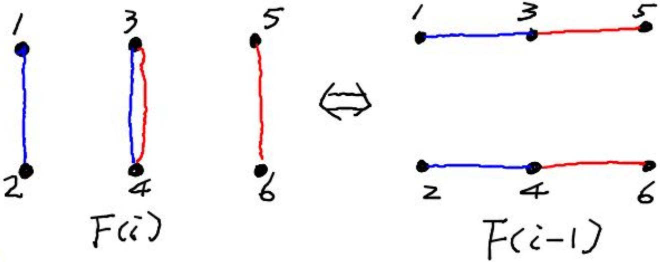

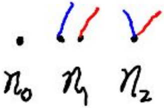

$$\frac{|F(i, n_2)|}{|F(i-1, n_2)|} \approx \frac{n_2^2}{8i(\ell-i)^2} \approx \frac{n_2^2}{8i\ell^2}$$

The supercritical case

Estimate $|F(i)| = \sum_{n_2} |F(i, n_2)|$.

Key ingredients to complete the proof –

• $F(i, n_2)$ maximizes at some n_2^*. The value of n_2^* depends on i and ℓ and is around linear in n.

• The main contribution to the summation is from n_2 such that $|n_2 - n_2^*| = O(\sqrt{n_2^*})$.

• We can approximate $\sum_{n_2=n_2^*-K\sqrt{n_2^*}}^{n_2^*+K\sqrt{n_2^*}} |F(i, n_2)|$ by an integral.

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- $F(i, n_2)$ maximizes at some n_2^*. The value of n_2^* depends on i and ℓ and is around linear in n.
- The main contribution to the summation is from n_2 such that $|n_2 - n_2^*| = O(\sqrt{n_2^*})$.
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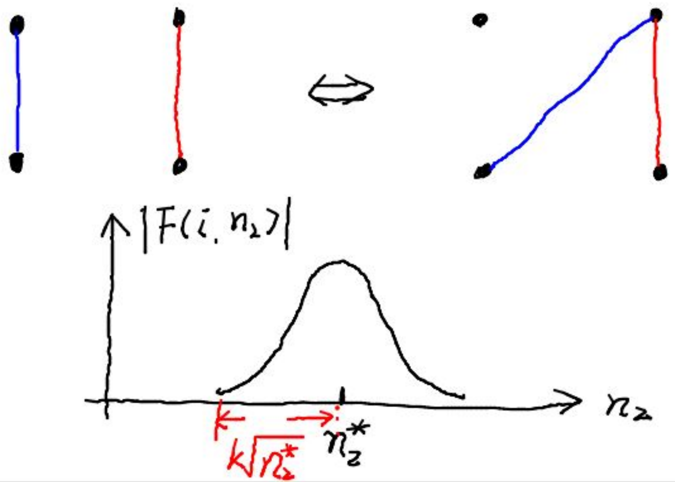
|F(i, n₂)|
n₂
n₂*
k√n₂*

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