Bijections for lattice paths between two boundaries

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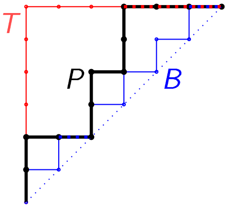
Joint work with Martin Rubey

CanaDAM 2013

Dyck paths



$t(P) = 3 \quad b(P) = 2$

For $P \in \mathcal{D}_n$ (Dyck paths with $2n$ steps), let

$$t(P) = \# \text{ of } E \text{ steps in common with } T$$
$$= \text{"height" of the last "peak"}$$
$$b(P) = \# \text{ of } E \text{ steps in common with } B$$
$$= \text{number of returns}$$

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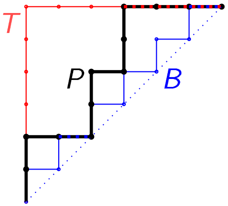

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Theorem (Deutsch ’98)

The joint distribution of the pair (t, b) *over* $\mathcal{D}_n$ *is symmetric, i.e.,*

$$\sum_{P \in \mathcal{D}_n} x^{t(P)} y^{b(P)} = \sum_{P \in \mathcal{D}_n} x^{b(P)} y^{t(P)}.$$

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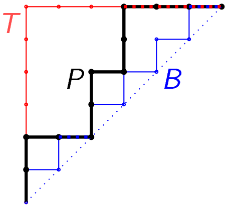

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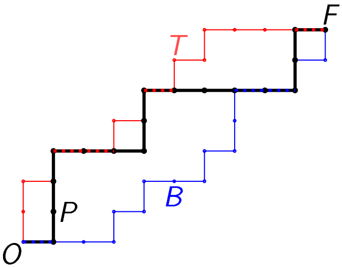
Proof 1 (Deutsch): Recursive bijection. Proof 2: Generating fcts.
Both proofs rely on the recursive structure of Dyck paths.

A generalization to arbitrary boundaries



$t(P) = 4 \quad b(P) = 3$

T and B paths from O to F with steps N and E, with T weakly above B

$P \in \mathcal{P}(T, B) =$ set of paths from O to F weakly between T and B

$t(P) = \#$ of E steps in common with T (top contacts of P)

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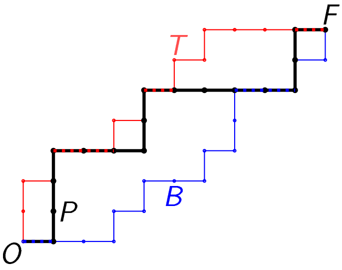

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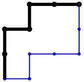
Theorem

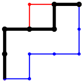
The joint distribution of (t,b) over $\mathcal{P}(T,B)$ is symmetric, i.e.,

$$\sum_{P\in\mathcal{P}(T,B)} x^{t(P)}y^{b(P)} = \sum_{P\in\mathcal{P}(T,B)} x^{b(P)}y^{t(P)}.$$

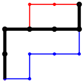
Example



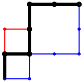
x^3



x^2



x



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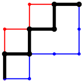

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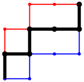
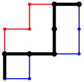
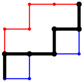


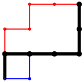
1



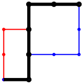
xy



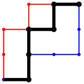
y



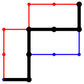
y^2



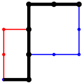
x^2y



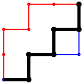
xy



y



xy^2



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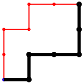

y^3

$$\sum_{P\in\mathcal{P}(T,B)} x^{t(P)}y^{b(P)} = x^3 + x^2y + xy^2 + y^3 + 2x^2 + 2xy + 2y^2 + 2x + 2y + 1$$

Proof

The known proofs for Dyck paths do not seem to generalize to arbitrary boundaries.

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We give an involution

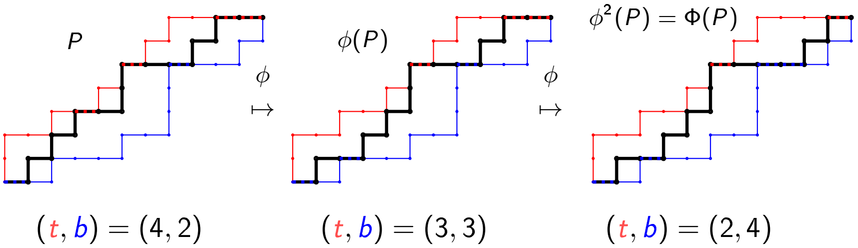
$$\Phi : \mathcal{P}(T, B) \to \mathcal{P}(T, B)$$

with the property $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$.

Idea: Given $P \in \mathcal{P}(T, B)$ with $t(P) > b(P)$, turn some of its top contacts into bottom contacts, one at a time.

Example

We define the involution Φ by iterating a map ϕ, which turns **one** top contact into one bottom contact.



From paths to words

To define $\phi(P)$, we first find the top contact that will be changed into a bottom contact.

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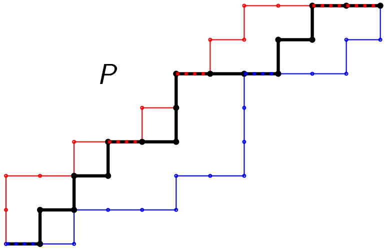
1. Record top and bottom contacts of P as a word $\mathbf{w}$ over $\{\mathbf{t}, \mathbf{b}\}$:



$\mathbf{w} = \mathbf{bttbtt}$

From paths to words

2. Having built **w**, select a top contact as follows:

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   - Draw a path with a step $(1,1)$ for each **t**, and a step $(1,-1)$ for each **b**.

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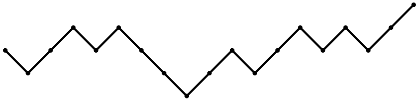

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   - Draw a path with a step $(1,1)$ for each **t**, and a step $(1,-1)$ for each **b**.
   - Match **t**'s and **b**'s that "face" each other in the path.

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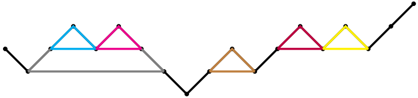

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   - Seleft the leftmost unmatched **t** as the top contact that will be changed.

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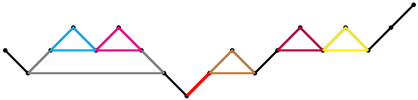

The map ϕ

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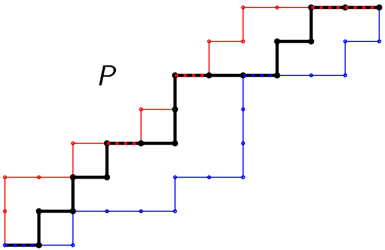

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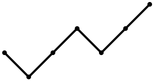

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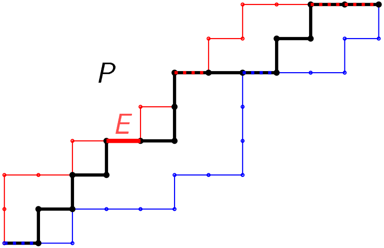

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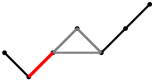

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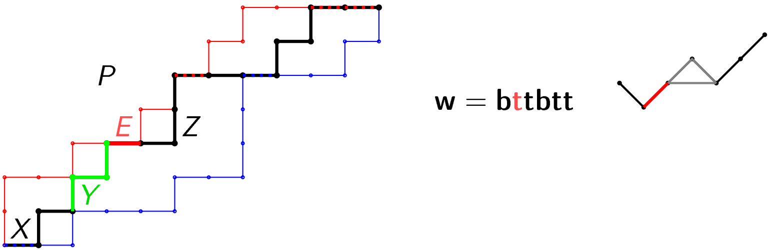
- Write $P = XYEZ$, where Y touches B only at its left endpoint.



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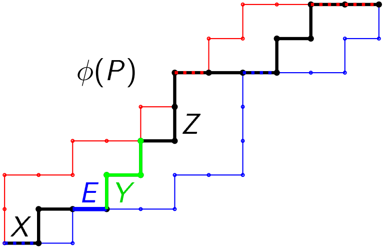
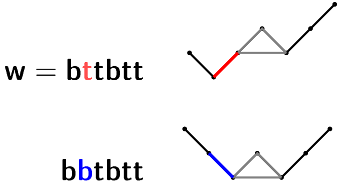
- Let $\phi(P) = XEYZ$.





The involution Φ

For $P \in \mathcal{P}(T, B)$ with $t(P) = e$ and $b(P) = f$, define

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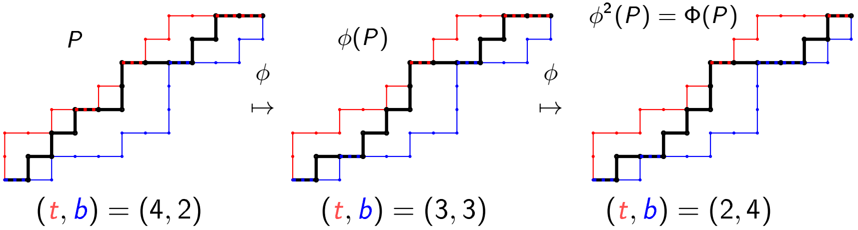
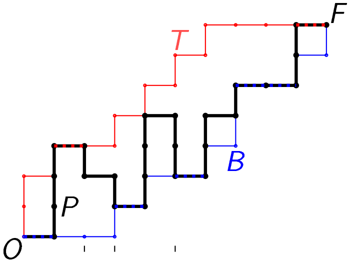


A generalization to paths with S steps



$\widetilde{\mathcal{P}}(T, B)$ = set of paths from O to F with steps N, E and S weakly between T and B.

For $P \in \widetilde{\mathcal{P}}(T, B)$, define $t(P)$ and $b(P)$ as before.
The *descent set* of P is the set of x-coordinates where S steps occur.

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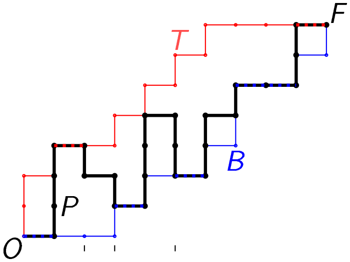

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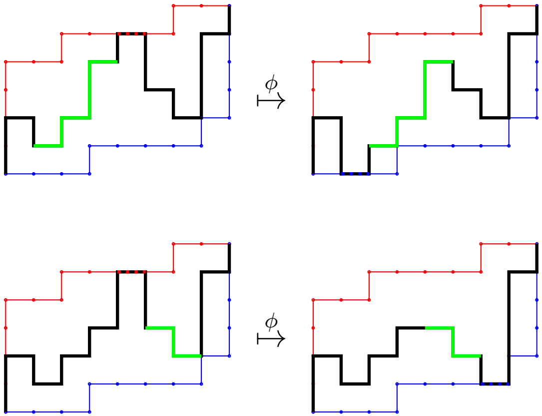
Theorem

There is an involution $\widetilde{\mathcal{P}}(T, B) \to \widetilde{\mathcal{P}}(T, B)$ *that switches the statistics* (t, b) *and preserves the descent set.*

A generalization: examples

The map ϕ for paths with S steps:



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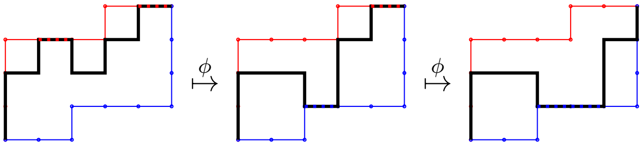
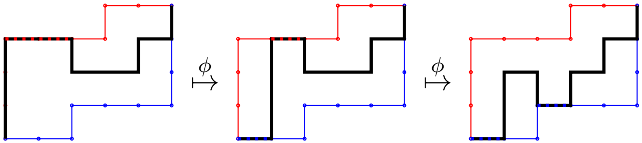
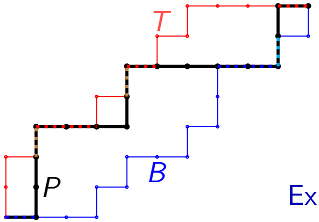




A related theorem



For $P \in \mathcal{P}(T, B)$, let

$\ell(P) = \#$ of N steps in common with T

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Example: $t(P) = 4$, $b(P) = 3$, $\ell(P) = 2$, $r(P) = 1$.

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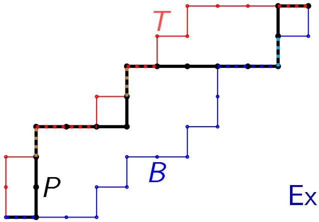

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Theorem

The pairs (b, ℓ) *and* (t, r) *have the same joint distribution over* $\mathcal{P}(T, B)$*, i.e.,*

$$\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} = \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{r(P)}.$$

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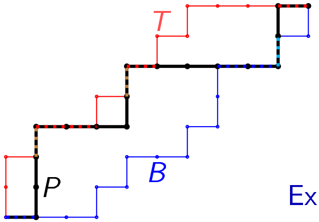

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The pairs (b, ℓ) and (t, r) have the same joint distribution over $\mathcal{P}(T, B)$, i.e.,

$$\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} = \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{r(P)}.$$

We do not know of a bijective proof similar to the previous one.

Proof idea

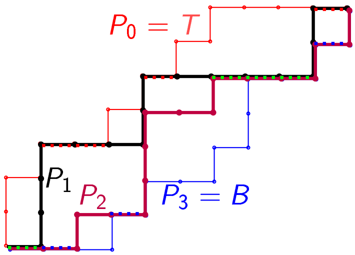
Both

$$\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} \quad \text{and} \quad \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{r(P)}$$

equal the Tutte polynomial of a *lattice path matroid*, as defined by Bonin–De Mier–Noy '03.

The statistics b and ℓ (t and r) are internal and external activities with respect to different linear orderings of the ground set.

k-fans of paths



$P_1, P_2, \ldots, P_k \in \mathcal{P}(T, B)$,
P_i weakly above P_{i+1} for all i.
Let $P_0 = T$, $P_{k+1} = B$.
For $0 \leq i \leq k$, let

$$h_i = \# \text{ of } E \text{ steps where } P_i \text{ and } P_{i+1} \text{ conincide}$$

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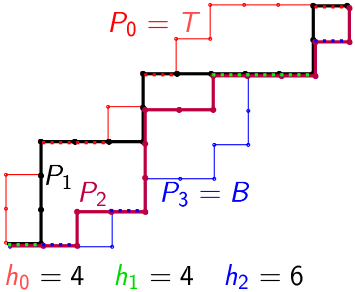

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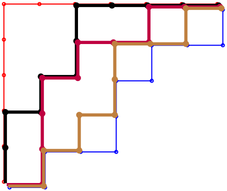
Theorem

The distribution of $(h_0, h_1, \ldots, h_k)$ over k-fans of paths as above is symmetric.

Connection to flagged SSYT

Let $T = NN \dots NEE \dots E$.



$h_i = \#\ E$ steps in $P_i \cap \mathcal{P}_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$

Connection to flagged SSYT

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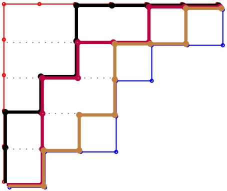
$u_1 = 2$
$u_2 = 2$
$u_3 = 1$
$u_4 = 1$



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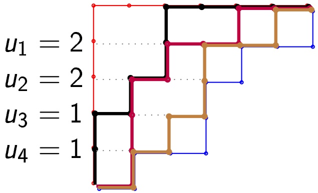

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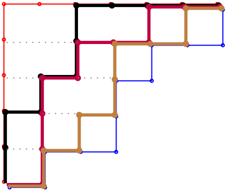

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1	1	2	2	3	4	≤ 4
2	3	3	4			≤ 5
4	5	6				≤ 6
5	6	7				≤ 7
8						≤ 8

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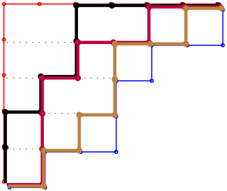

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Connection to flagged SSYT

Theorem

There is an explicit bijection between

- *k-fans of paths in $\mathcal{P}(T, B)$ with statistics h_i and u_j, and*
- *k-flagged SSYT of shape λ and weight*

$$(\lambda_1 - h_0, \lambda_1 - h_1, \ldots, \lambda_1 - h_k,\ u_1, u_2, \ldots, u_r).$$

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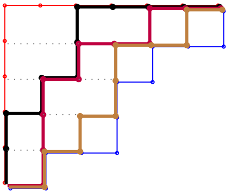

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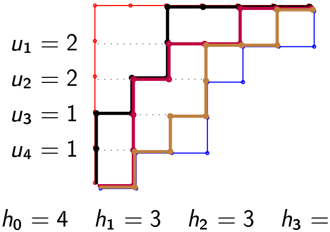

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The bijection uses a variation of *jeu de taquin*.

Connection to k-triangulations

Theorem (conjectured by C. Nicolás '09)

The joint distribution of the degrees of $k+1$ consecutive vertices in a k-triangulation of a convex n-gon equals the distribution of $(h_0, h_1, \ldots, h_k)$ over k-fans of Dyck paths of semilength $n-2k$.

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The proof uses the previous theorem in the special case of Dyck paths, together with a bijection of Serrano–Stump between k-triangulations and k-flagged SSYT.