Graceful Labellings of Triangular Cacti

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Graceful labelling

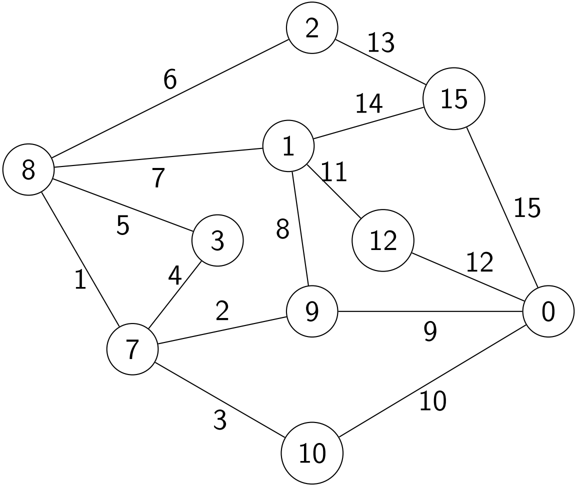
For a graph $G = (V, E)$ on m edges, an injection $f : V \mapsto \{0, 1, 2, \ldots, m\}$ with the property that for all edges $uv \in E$, $\{|f(u) - f(v)|\} = \{1, 2, 3, \ldots, m\}$ is a graceful labelling of G.

(Coined as a β-valuation by Rosa in 1967; graceful by Golomb in 1972.)

For a graph $G = (V, E)$ on m edges, an injection $f : V \mapsto \{0, 1, 2, \ldots, m + 1\}$ with the property that for all edges $uv \in E$, $\{|f(u) - f(v)|\} = \{1, 2, 3, \ldots, m - 1, m + 1\}$ is a near graceful labelling of G.

A graph is (near) graceful if it admits a (near) graceful labelling.

An Example



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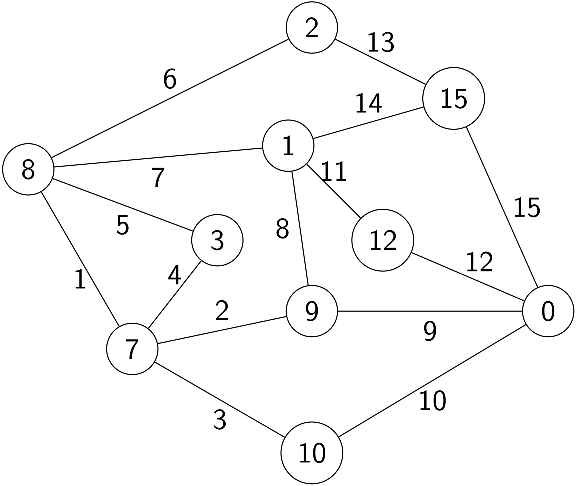

What graphs can be gracefully labelled?

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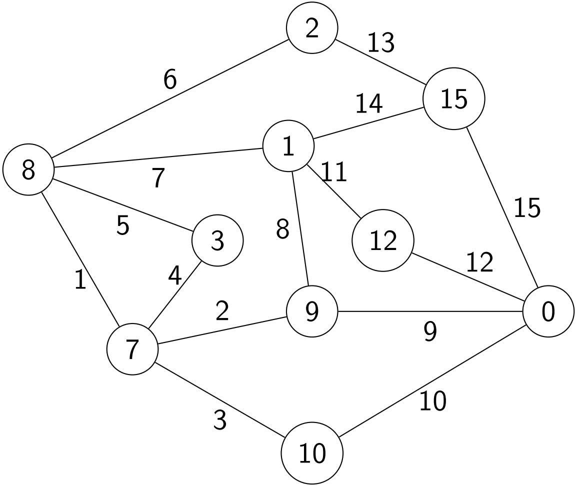

What graphs can be gracefully labelled? Lots.

Some conjectures

Ringel-Kotzig Conjecture

All trees are graceful.

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A triangular cactus is a connected graph all of whose blocks are triangles.

Rosa's Conjecture

All triangular cacti with $t \equiv 0, 1 \bmod 4$ blocks are graceful, and those with $t \equiv 2, 3 \bmod 4$ are near graceful.

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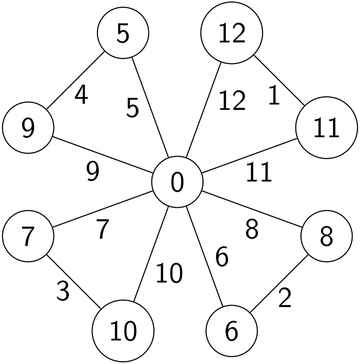
Gallian suggests this is "hopelessly difficult."

Windmills

A regular Dutch windmill is a triangular cactus in which all blocks have a common vertex that we will call the central vertex. The blocks will be called vanes.

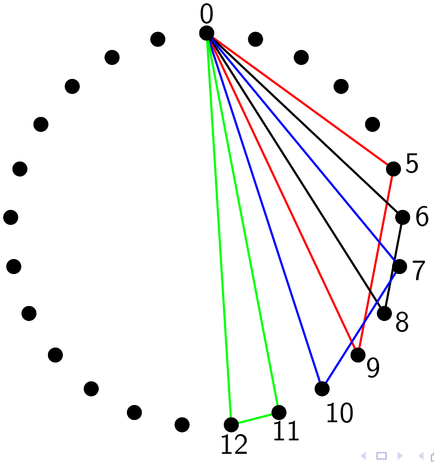


Bermond (1979) showed that all regular Dutch windmills are graceful or near graceful.

Who cares?

Theorem

If G is graceful with m edges, then K_{2m+1} is G-decomposable.



Skolem sequences

A Skolem sequence of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions

1. for every $k \in \{1, 2, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$, and
2. if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

A Skolem sequence of order 4:

$$\frac{4}{1}\ \frac{2}{2}\ \frac{3}{3}\ \frac{2}{4}\ \frac{4}{5}\ \frac{3}{6}\ \frac{1}{7}\ \frac{1}{8}$$

Skolem sequences, again.

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Sometimes we just write the pairs of indices.

$$(1,5),(2,4),(3,6),(7,8)$$

These pairs are useful.

$$\begin{array}{cc} (a_i, b_i) & \\ (7,8) & \\ (2,4) & \rightarrow \\ (3,6) & \\ (1,5) & \end{array}$$

Skolem sequences, again.

A Skolem sequence of order 4:

$$\begin{array}{cccccccc} 4 & 2 & 3 & 2 & 4 & 3 & 1 & 1 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

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$$\begin{array}{ccc} (a_i, b_i) & & (i, a_i+n, b_i+n) \\ (7,8) & & (1,11,12) \\ (2,4) & \rightarrow & (2,6,8) \\ (3,6) & & (3,7,10) \\ (1,5) & & (4,5,9) \end{array}$$

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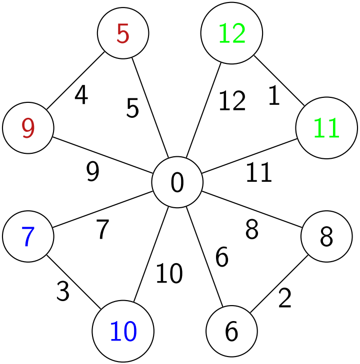
$$\begin{array}{ccccc} (a_i,b_i) & & (i,a_i+n,b_i+n) & & (0,a_i+n,b_i+n) \\ (7,8) & & (1,11,12) & & (0,11,12) \\ (2,4) & \rightarrow & (2,6,8) & \rightarrow & (0,6,8) \\ (3,6) & & (3,7,10) & & (0,7,10) \\ (1,5) & & (4,5,9) & & (0,5,9) \end{array}$$

Graceful labelling from Skolem sequences

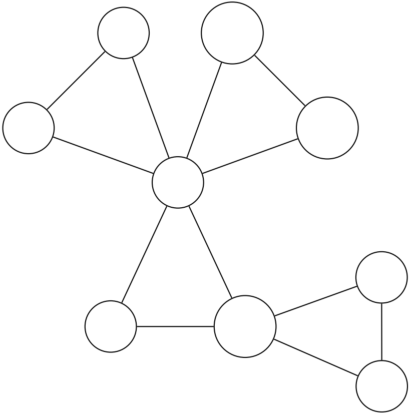
4	2	3	2	4	3	1	1
1	2	3	4	5	6	7	8

$(0, 11, 12)$, $(0, 6, 8)$, $(0, 7, 10)$, $(0, 5, 9)$



Windmills with a pendant triangle

Now something a little harder...



Back to Skolem sequences

A number i ($1 \leq i \leq n$) is a pivot of a Skolem sequence if $b_i + i \leq 2n$.

$$\begin{array}{cccccccc} 4 & \overbrace{2 & 3 & 2} & 4 & 3 & 1 & 1 \\ \hline 1 & 2 & 3 & \underbrace{4 & 5 & 6} & 7 & 8 \end{array}$$

We see 2 is a pivot.

$$\begin{array}{ccc} (a_i, b_i) & & (0, a_i + n, b_i + n) \\ (7, 8) & & (0, 11, 12) \\ (2, 4) & \rightarrow & (0, 6, 8) \\ (3, 6) & & (0, 7, 10) \\ (1, 5) & & (0, 5, 9) \end{array}$$

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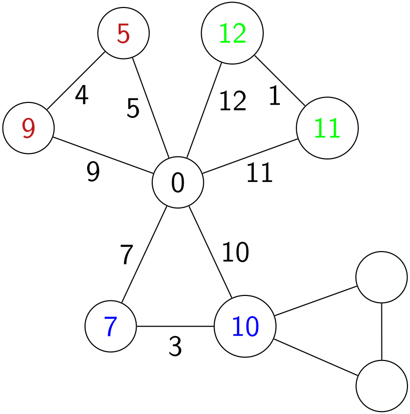
Replace the pivot's triple $(0, a_j + n, b_j + n)$ with $(j, a_j + j + n, b_j + j + n)$.

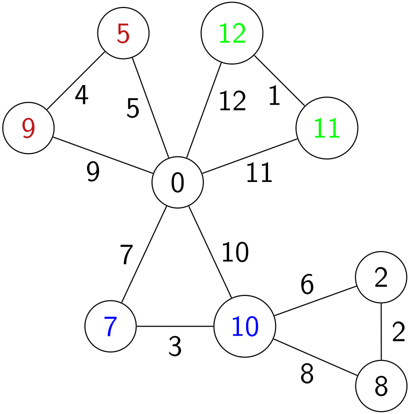
Back to the windmills...

$(0, 11, 12), (0, 7, 10), (0, 5, 9), (0, 6, 8)$

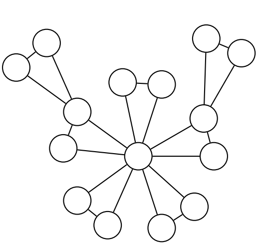


Back to the windmills...

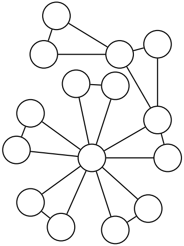
$(0, 11, 12)$, $(0, 7, 10)$, $(0, 5, 9)$, ~~$(0, 6, 8)$~~ $\rightarrow (2, 8, 10)$



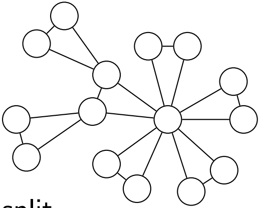
Windmills with two pendant triangles



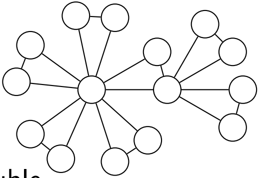
independent



stacked



split

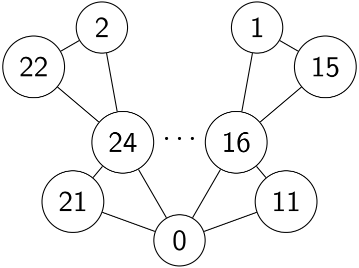


double

Independent

$$\frac{4}{1}\ \frac{8}{2}\ \frac{5}{3}\ \frac{7}{4}\ \frac{4}{5}\ \frac{1}{6}\ \underbrace{\frac{1}{7}\ \frac{5}{8}}\ \frac{6}{9}\ \frac{8}{10}\ \frac{7}{11}\ \frac{2}{12}\ \frac{3}{13}\ \underbrace{\frac{2}{14}\ \frac{6}{15}\ \frac{3}{16}}$$

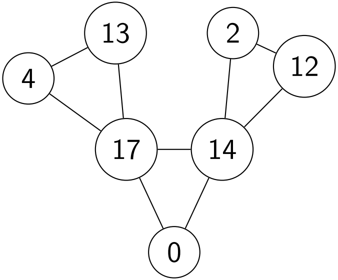
$$\begin{array}{ccccc} (6,7) & \to & (0,14,15) & \to & (1,15,16) \\ (12,14) & \to & (0,20,22) & \to & (2,22,24) \end{array}$$



Split

4	2	7	2	4	3	8	6	3	7	5	1	1	6	8	5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

$$
\begin{array}{ccccc}
(1,5) & \rightarrow & (0,9,13) & \rightarrow & (4,13,17) \\
(2,4) & \rightarrow & (0,10,12) & \rightarrow & (2,12,14)
\end{array}
$$



Some results

Theorem

A Dutch windmill with one pendant triangle is either graceful or near graceful.

Theorem

A Dutch windmill with two pendant triangles is either graceful or near graceful.

A note on method

A Langford sequence of order n and defect d is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2n})$ of $2n$ integers satisfying the conditions

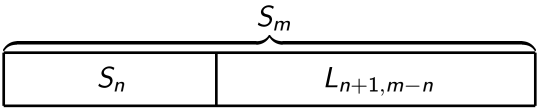
1. for every $k \in \{d, d+1, \ldots, d+n-1\}$ there exist exactly two elements $\ell_i, \ell_j \in L$ such that $\ell_i = \ell_j = k$, and
2. if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

A Langford sequence of order 4 and defect 2.

$$\frac{5}{1}\ \frac{2}{2}\ \frac{4}{3}\ \frac{2}{4}\ \frac{3}{5}\ \frac{5}{6}\ \frac{4}{7}\ \frac{3}{8}$$

A way forward

Let S_n be a Skolem sequence, and $L_{n+1,m-n}$ be a Langford sequence of order $m-n$ with defect $n+1$.



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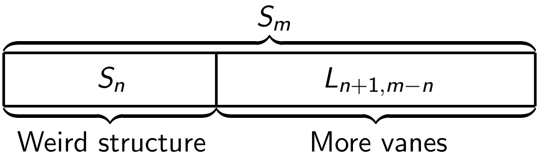


Three pendent triangles

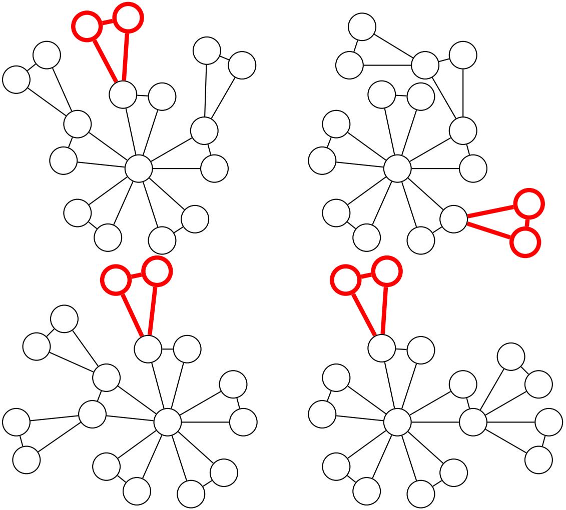
We can push this problem a littler farther. To do this, we use a combination of our previous methods.

Unfortunately, there are 11 cases to consider.

Some follow directly from the previous constructions, or from new Skolem sequences.

One is a big problem.

Low hanging fruit

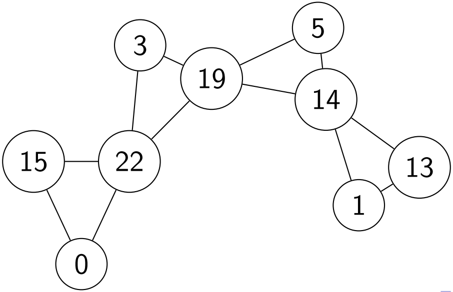


A little harder...

But there are seven more cases! Some are similar.

$$\frac{5}{1}\ \frac{8}{2}\ \frac{6}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{5}{6}\ \frac{7}{7}\ \frac{3}{8}\ \frac{6}{9}\ \frac{8}{10}\ \frac{3}{11}\ \frac{4}{12}\ \frac{2}{13}\ \frac{7}{14}\ \frac{2}{15}\ \frac{4}{16}$$

$$\begin{array}{ccccc} (4,5) & \to & (0,12,13) & \to & (1,13,14) \\ (8,11) & \to & (0,16,19) & \to & (3,19,22) \\ (1,6) & \to & (0,9,14) & \to & (5,14,19) \end{array}$$

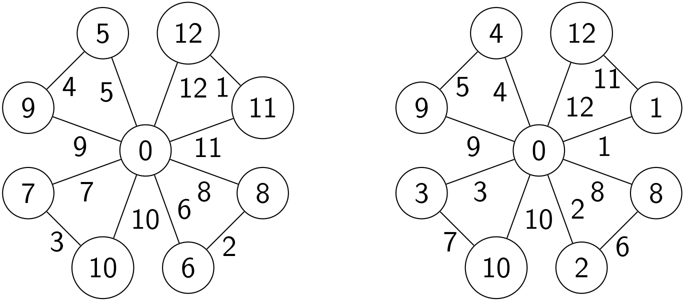


A new trick

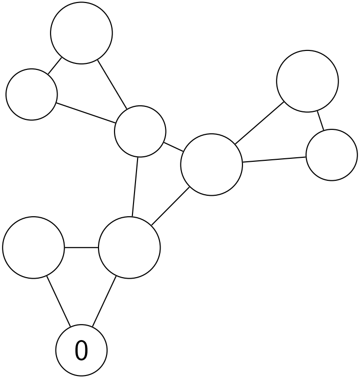
$$\begin{array}{cccccccc} 4 & 2 & 3 & 2 & 4 & 3 & 1 & 1 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$(0, a_i + n, b_i + n)$$
$$(0, 11, 12)$$
$$(0, 6, 8)$$
$$(0, 7, 10)$$
$$(0, 5, 9)$$

$$(0, i, b_i + n)$$
$$(0, 1, 12)$$
$$(0, 2, 8)$$
$$(0, 3, 10)$$
$$(0, 4, 9)$$



One is much harder

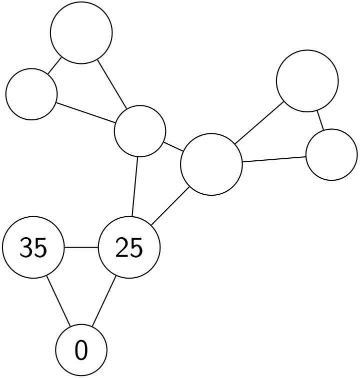


A bad pivoting structure. We use our trick, and some extra "shifting", beyond pivoting. Instead of shifting $(0, a_i + n, b_i + n)$ to $(i, a_i + i + n, b_i + i + n)$, we could shift it to $(k, a_i + k + n, b_i + k + n)$, for any k.

One is much harder

$$\frac{4}{1}\ \frac{2}{2}\ \frac{3}{3}\ \frac{2}{4}\ \frac{4}{5}\ \frac{3}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{10}{9}\ \frac{8}{10}\ \frac{15}{11}\ \frac{16}{12}\ \frac{14}{13}\ \frac{11}{14}\ \frac{6}{15}\ \frac{7}{16}\ \frac{13}{17}\ \frac{8}{18}\ \frac{10}{19}\ \frac{12}{20}\ \frac{6}{21}\ \frac{9}{22}\ \frac{7}{23}\ \frac{5}{24}\ \frac{11}{25}\ \frac{15}{26}\ \frac{14}{27}\ \frac{16}{28}\ \frac{5}{29}\ \frac{13}{30}\ \frac{9}{31}\ \frac{12}{32}$$

$(0, 23, 24)$
$(0, 18, 20)$
$(0, 19, 22)$
$(0, 25, 35)$
$(0, 27, 42)$



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$(0, 23, 24)$
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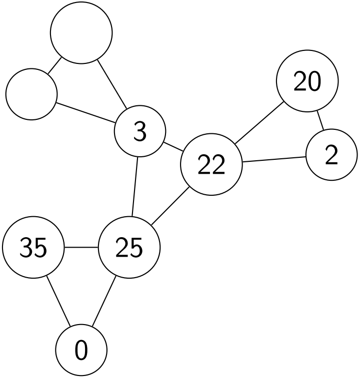
~~$(0, 19, 22)$~~ $\rightarrow$ $(3, 22, 25)$
$(0, 25, 35)$
$(0, 27, 42)$



One is much harder

$$\frac{4}{1}\ \frac{2}{2}\ \frac{3}{3}\ \frac{2}{4}\ \frac{4}{5}\ \frac{3}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{10}{9}\ \frac{8}{10}\ \frac{15}{11}\ \frac{16}{12}\ \frac{14}{13}\ \frac{11}{14}\ \frac{6}{15}\ \frac{7}{16}\ \frac{13}{17}\ \frac{8}{18}\ \frac{10}{19}\ \frac{12}{20}\ \frac{6}{21}\ \frac{9}{22}\ \frac{7}{23}\ \frac{5}{24}\ \frac{11}{25}\ \frac{15}{26}\ \frac{14}{27}\ \frac{16}{28}\ \frac{5}{29}\ \frac{13}{30}\ \frac{9}{31}\ \frac{12}{32}$$

~~$(0, 23, 24)$~~ $\rightarrow$ $(0, 1, 24)$

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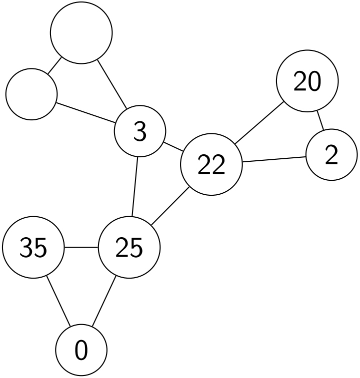

One is much harder

4	2	3	2	4	3	1	1	10	8	15	16	14	11	6	7	13	8	10	12	6	9	7	5	11	15	14	16	5	13	9	12
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32

~~$(0, 23, 24)$~~ $\rightarrow$ ~~$(0, 1, 24)$~~ $\rightarrow$ $(3, 4, 27)$

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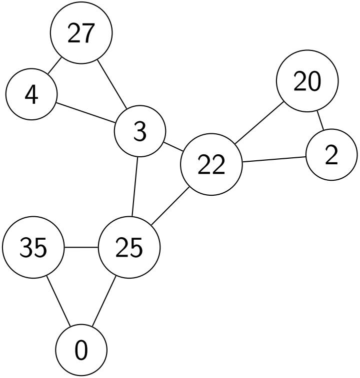

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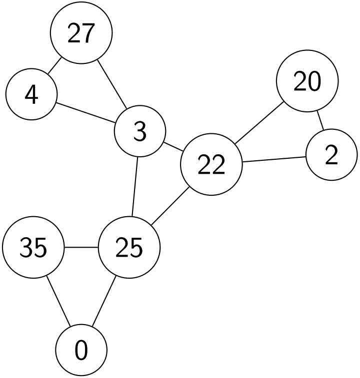

Conclusions and Open Questions

Theorem

A Dutch windmill with three pendant triangles is either graceful or near graceful.

1. Define $S(n)$ to be the number of triangular cacti of order n which can be labelled using Skolem sequences and their pivots. Define $T(n)$ to be the number of triangular cacti of order n. What is $\lim_{n\to\infty} \frac{S(n)}{T(n)}$?
2. Can we use the Langford sequence construction to gracefully label other graphs?

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Thank you!