Bipartite 2-factorisations of complete multipartite graphs

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The Oberwolfach problem

The Oberwolfach problem was posed by Ringel in the 1960s.
At the Conference center in Oberwolfach, Germany

The Oberwolfach problem was originally motivated as a seating problem:

The Oberwolfach problem

Given n attendees at a conference with t circular tables each of which which seat a_i, $i = 1, \ldots, t$ people $\left(\sum_{i=1}^{t} a_i = n\right)$.

Find a seating arrangement so that every person sits next to each other person around a table exactly once over the r days of the conference.

Factors

Definition

A *k*-factor of a graph G is a k-regular spanning subgraph of G.

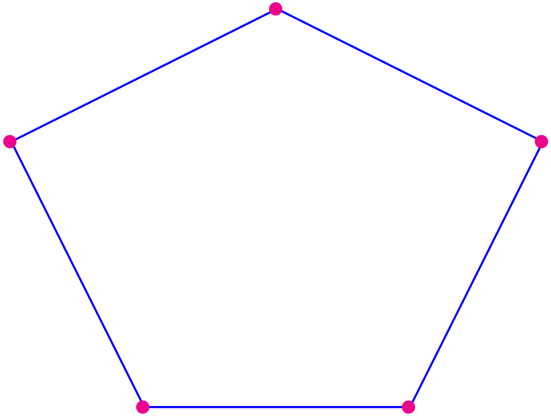
Definition

Given a factor F, an $F-$Factorisation of a graph G is a decomposition of the edges of G into copies of F.

Definition

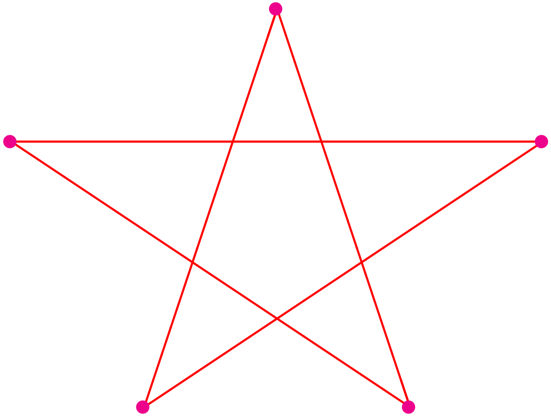
Given a set of factors $\mathcal{F}$, an $\mathcal{F}-$Factorisation of a graph G is a decomposition of the edges of G into copies of factors $F \in \mathcal{F}$.

Example $n = 5$



A 2−Factor of K_5

Example $n = 5$



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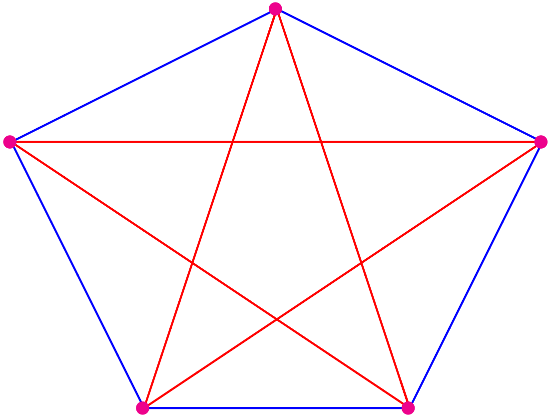


A $2-$Factorisation of K_5

The Oberwolfach problem

When n is odd, the Oberwolfach problem $\mathrm{OP}(F)$ asks for a factorisation of K_n into a specified $2-$factor F of order n.

$$\left(r = \frac{n-1}{2}\right)$$

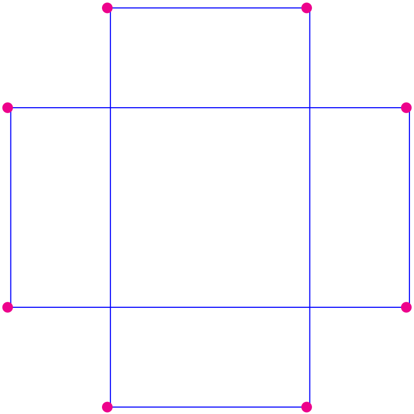
When n is even, the Oberwolfach problem $\mathrm{OP}(F)$ asks for a factorisation of $K_n - I$ into a specified $2-$factor F of order n.
Where $K_n - I$ is the complete graph on n vertices with the edges of a 1-factor removed.

$$\left(r = \frac{n-2}{2}\right)$$

We will use the notation $[m_1, m_2, \ldots, m_t]$ to denote the 2-regular graph consisting of t (vertex-disjoint) cycles of lengths $m_1, m_2, \ldots, m_t$.

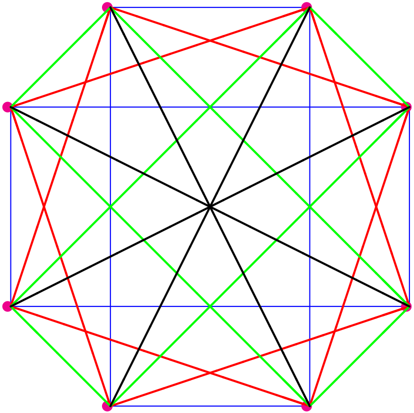
The Oberwolfach problem can be thought of as a generalisation of Kirkman Triple Systems, which are the case $F = [3, 3, \ldots, 3]$.

Example $n = 8, \quad F = [4, 4]$



A $[4, 4]-$Factor of K_8

Example $n = 8$, $F = [4, 4]$



An $[4, 4]-$Factorisation of K_8 with a $1-$factor remaining

What is known about $\mathrm{OP}(F)$

It is known that there is no solution to $\mathrm{OP}(F)$ for

$$F \in \{[3,3],[4,5],[3,3,5],[3,3,3,3]\},$$

A solution is known for all other instances with $n \leq 40$.
Deza, Franek, Hua, Meszka, Rosa (2010), Adams & Bryant (2006), Franek & Rosa (2000), Bolstad (1990), Huang, Kotzig & A. Rosa (1979).

The case where all the cycles in F are of the same length has been solved.
Govzdjak (1997), Alspach & Häggkvist (1985), Alspach, Schellenberg, Stinson & D. Wagner (1989), Hoffman & Schellenberg (1991), Huang, Kotzig & A. Rosa (1979), Ray-Chaudhuri & Wilson (1971).

Generalise to $\mathrm{OP}(F_1, \ldots, F_t)$

Given 2−factors $F_1, F_2, \ldots, F_t$ order n and non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_t = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ \frac{n-2}{2} & n \text{ even} \end{cases}$$

Find a 2-factorisation of K_n, or $K_n - I$ if n is even, in which there are exactly α_i 2-factors isomorphic to F_i for $i = 1, 2, \ldots, t$.

Generalise to other Graphs G, $\mathrm{OP}(G; F_1, \ldots, F_t)$

We can also consider Factorisations of other graphs G.

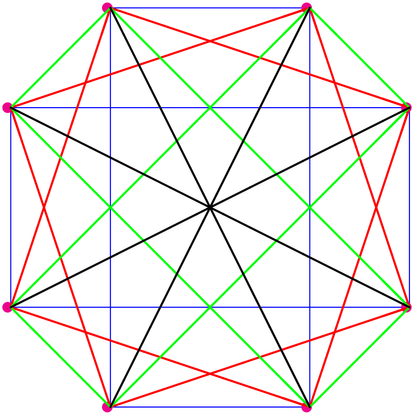
We write $\mathrm{OP}(G; F_1, \ldots, F_t)$ to denote a factorisation of G into factors each isomorphic to one of the F_i.

Of particular interest is the case when $G = K_{n^r}$, the multipartite complete graph with r parts of size n.

Given a bipartite factor F of order nr, we consider the problem of finding an $F-$factorisation of K_{n^r}.

i.e. A solution to $\mathrm{OP}(K_{n^r}; F)$.

Example $G = K_{2^4}$, $F = [4, 4]$



A $[4, 4]-$Factorisation of K_{2^4}

Generalised OP and Häggkvist

Theorem (Häggkvist (1985))

Let $n \equiv 2 \bmod 4$, and $F_1, \ldots, F_t$ be bipartite $2-$factors of order n then $\mathrm{OP}(F_1, \ldots, F_t)$ has solution, with an even number of factors isomorphic to each F_i.

Corollary ($t = 1$)

Let $n \equiv 2 \bmod 4$, and F be a bipartite $2-$factor of order n then $\mathrm{OP}(F)$ has solution.

Interpreting the 1-factor as parts of size 2...

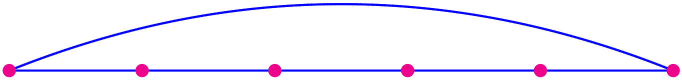
Corollary

Let $n \equiv 2 \bmod 4$, and F be a bipartite $2-$factor of order $2n$ then $\mathrm{OP}(K_{2^n}; F)$ has solution.

Doubling

For any given graph G, the graph $G^{(2)}$ is defined by
$V(G^{(2)}) = V(G) \times \mathbb{Z}_2$,
$E(G^{(2)}) = \{\{(x,a),(y,b)\} : \{x,y\} \in E(G),\ a,b \in \mathbb{Z}_2\}$.

C



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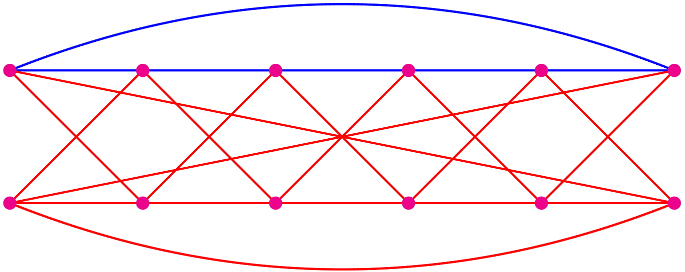
$C^{(2)}$

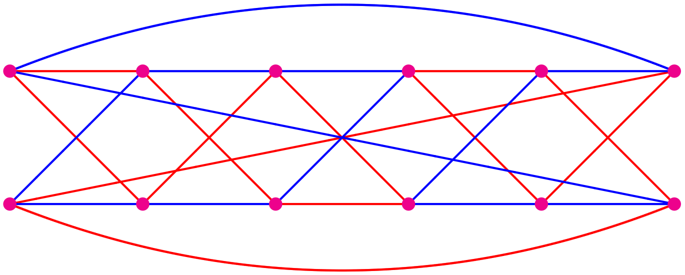


Factoring $C_n^{(2)}$

Lemma (Häggkvist (1985))

For any $m > 1$ and for each bipartite 2-regular graph F of order $2m$, there exists a 2-factorisation of $C_m^{(2)}$ in which each 2-factor is isomorphic to F.

$C_m^{(2)}$



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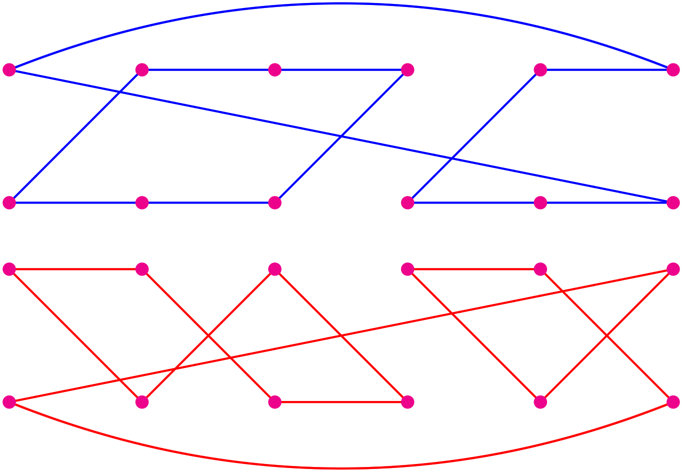


Multipartite Graphs - One bipartite Factor F

We wish to consider factorisations of the complete multipartite graph K_{n^r} into a single biparite $2-$factor F.

Ths is the multipartite case of the original Oberwolfach problem.

Necessary Conditions

In order for the complete multipartite graph K_{n^r} to have a factorisation into 2−factors $F_1, \ldots, F_t$ every vertex must be of even degree.

i.e. $n(r-1)$ is even.

Now we only have one bipartite factor F, of even order nr,

But $n(r-1)$ is also even,so:

Theorem

In order for the complete multipartite graph K_{n^r}, $r \geq 2$, to have a factorisation into a single bipartite 2−factor, n must be even.

What is known

Theorem (Auerbach and Laskar (1976))

A complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.

Theorem (Liu (2003))

The complete multipartite graph K_{n^r}, $r \geq 2$, has a $2-$factorisation into $2-$factors composed of $k-$cycles if and only if $k \mid rn$, $(r-1)n$ is even, further k is even when $r = 2$, and $(k, r, n) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}$.

What is known

Theorem (Piotrowski (1991))

If F is a bipartite $2-$regular graph of order $2n$, then the complete bipartite graph, K_{n^2} has a $2-$factorisation into F except when $n = 6$ and $F = [6, 6]$.

Theorem (Häggkvist (1985), Bryant, Danziger (2010))

If F is a bipartite $2-$regular graph of order $2r$, then the complete multipartite graph K_{2^r} has a $2-$factorisation into F.

These deal with factorisations of K_{n^r} in the cases $r = 2$ and $n = 2$ respectively.

Ciculant Graphs

A Cayley graph on a cyclic group is called a circulant graph.
We will always use vertex set $\mathbb{Z}_n$.

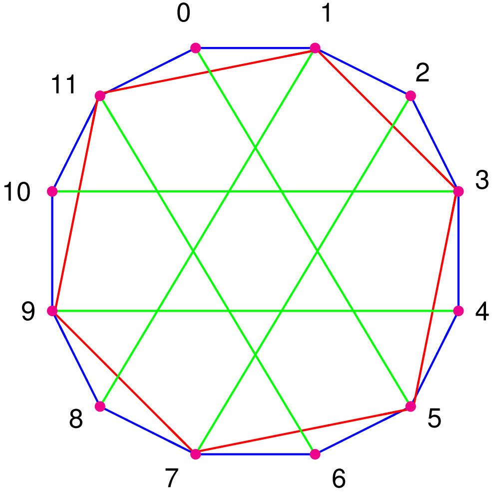
The length of an edge $\{x, y\}$ in a graph is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$

We denote by $\langle S \rangle_n$ the graph with vertex set $\mathbb{Z}_n$ and edge set the edges of length s for each $s \in S$.

We call $\{\{x, x + s\} : x = 0, 2, \ldots, n - 2\}$ the even edges of length s.
We call $\{\{x, x + s\} : x = 1, 3, \ldots, n - 1\}$ the odd edges of length s.

If we wish to include in our graph only the even edges of length s then we give s the superscript e.
If we wish to include only the odd edges of length s then we give s the superscript o.

Example $\langle\{1, 2^{\mathrm{o}}, 5^{\mathrm{e}}\}\rangle_{12}$ on $\mathbb{Z}_{12}$



Two very useful Theorems

Theorem (Bermond, Favaron, Mahéo, (1989))

Every connected 4-regular Cayley graph on a finite abelian group has a Hamiltonian decomposition.

Theorem (Dean (2006))

Every 6-regular Cayley graph on a cyclic group which has a generator of the group in its connection set has a Hamiltonian decomposition.

Setting up

Note that when $n = 2m$

$$K_{(2m)^r} \cong K_{m^r}^{(2)}$$

Also

$$K_{m^r} \cong \left\langle \left\{1, 2, \ldots, \frac{rm}{2}\right\} \setminus \left\{r, 2r, \ldots, \frac{m-1}{2} r\right\} \right\rangle_{rm}.$$

And

$$\langle\{1, 3^e\}\rangle_{rm} \cong \left\langle \left\{1, \frac{rm}{2}\right\} \right\rangle_{rm}$$

.

Factoring K_{m^r}

Lemma

For each even $r \geq 4$ and each odd $m \geq 1$, except $(r, m) = (4, 1)$, there is a factorisation of K_{m^r} into $\frac{(r-1)m-3}{2}$ Hamilton cycles and a copy of $\langle\{1, 3^e\}\rangle_{rm}$.

Auerbach and Laskar's result proved that a there is a Hamiltonian decomposition of the complete multipartite graph, K_{n^r} if and only if it is regular of even degree. In particular, this implies that either r is even or n is odd.

This Lemma shows that there is a Hamiltonain decomposition of $K_{n^r} - I$, where I is a 1−factor, when n is even and r is odd.

To see this note that $\langle\{1\}\rangle_{rm}$ is a Hamiltonian cycle and $\langle\{3^e\}\rangle_{rm}$ is a 1−factor.

Aside

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Lemma

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Theorem (Bryant, Danziger, Dean (2012))

Let $n \equiv 0 \bmod 4$ with $n \geq 12$. For each bipartite 2-regular graph F of order n, there is a factorisation of $\langle\{1, 3^e\}\rangle^{(2)}_{n/2}$ into three copies of F; except possibly when $F \in \{[6^{4x+2}], [4, 6^{4x+2}] \mid x \geq 0\}$.

Corollary (Corollary 1)

Let $r \geq 4$ be even and $m \geq 1$ be odd, and let F be a bipartite 2−regular graph of order $2mr$. If $F \notin \{[6^{4x+2}], [4, 6^{4x+2}] \mid x \geq 0\}$, then there is an $F-$factorisation of $K_{(2m)^r}$.

The Case $[4, 6^{4x+2}]$.

Lemma

For each even $r \geq 4$ and each odd $m \geq 3$ such that $rm \equiv 8 \bmod 12$, there is a factorisation of K_{m^r} into $\frac{(r-1)m-5}{2}$ Hamilton cycles and a copy of $\langle\{1, 2, 3^e\}\rangle_{rm}$.

Lemma

For each $x \geq 1$, there is a factorisation of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12x+8}$ into five copies of $[4, 6^{4x+2}]$.

Corollary (Corollary 2)

Let $r \geq 4$ be even and $m \geq 3$ be odd. If $rm \equiv 8 \bmod 12$ there is a $[4, 6^{4x+2}]-$factorisation of $K_{(2m)^r}$ for every $x \geq 1$.

Necessary Conditions are Sufficient

Theorem (Bryant, Danziger, Pettersson (2013))

If F is a bipartite 2-regular graph of order rn, then there exists a 2-factorisation of K_{n^r}, $r \geq 2$, into F if and only if n is even; except that there is no 2-factorisation of $K_{6,6}$ into $[6,6]$.

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If m is even or r is odd, then K_{m^r} has even degree, and hence has a Hamilton decomposition by Auerbach and Laskar's result.

Setting $n = 2m$ gives $K_{n^r} \cong K_{m^r}^{(2)}$ and we can complete the proof using Häggkvist's doubling.

$n = 2$ and $r = 2$ are done above.

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This leaves the case $m \geq 3$ is odd and $r \geq 4$ is even.

If $F \notin \{[6^{4x+2}], [4, 6^{4x+2}], x \geq 0$, then there is an $F-$factorisation of $K_{(2m)^r}$ by Corollary 1.

The case $F = [6^{4x+2}]$ is covered by Liu's result.

When $F = [4, 6^{4x+2}]$ for some $x \geq 1$, this is covered by Corollary 2. □

The End

Thank You