Max-Point-Tolerance Graphs

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Minisymposium on **Geometric Representations of Graphs**
CanaDAM (June 11, 2013)

Outline

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Intersection Graphs

Definition

For a collection of sets $\mathbb{S} = \{S_0, ..., S_{n-1}\}$ the intersection graph of $\mathbb{S}$ has vertex set $\mathbb{S}$ and edge set $\{S_iS_j : i, j \in \{0, ..., n-1\}, i \neq j, \text{ and } S_i \cap S_j \neq \emptyset\}$

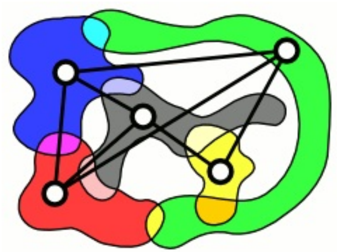
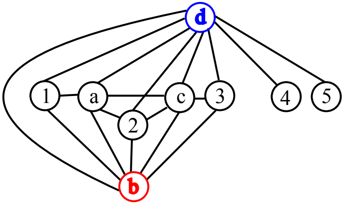
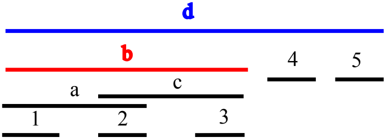


Figure : http://upload.wikimedia.org/wikipedia/commons/e/e9/Intersection_graph.gif

Interval Intersection Graph Classes

- **Interval**: intersection graphs of intervals of $\mathcal{R}$.
- **Tolerance**: interval graphs where each pair of intervals tolerate intersections up to $\min\{t_u, t_v\}$ without corresponding to edges.
- **Max-Tolerance**: interval graphs where each pair of intervals tolerate intersections up to $\max\{t_u, t_v\}$ without corresponding to edges.

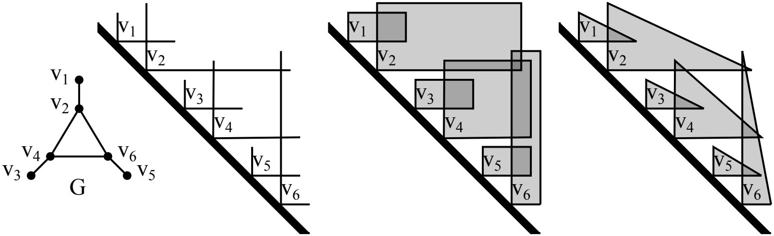




Geometric Graph Classes (all in the plane)

- **Rectangle**: intersection graphs of axis-aligned rectangles.
- **Right-Triangle**: intersection graphs of axis-aligned right triangles.
- **L**: intersection graphs of axis-aligned L-shapes.
- **Segment**: intersection graphs of line segments.
- **2-DIR**: intersection graphs of vertical and horizontal line segments.
- Semi-square: isosceles right-triangle = max-tolerance

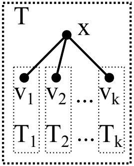
[M. Kaufmann, J. Kratochvil, K.A. Lehmann, A.R. Subramanian, SODA 2006].

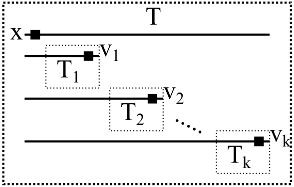


Max Point Tolerance (MPT).

Definition

$G = (V, E)$ is MPT when there exists pointed intervals $\{(I_v, p_v)\}_{v \in V}$ such that $uv \in E$ iff $\{p_u, p_v\} \subseteq I_u \cap I_v$.





[D. Catanzaro, B.V. Halldórsson, M. Labbé 2012]:

- At most n^2 maximal cliques; i.e., polytime algorithm for maximum clique.
- Weighted Clique Cover is NP-complete.

In this talk

Graph Class relationships:

- MPT includes interval graphs, complete bipartite graphs, and outerplanar graphs.
- MPT is included in: L, right-triangle, and rectangle.

Characterizations: (the following are equivalent)

- G is a max-point-tolerance graph.
- G is linear L = linear rectangle = linear right-triangle.
- G has a specific four point vertex ordering condition.
- G is a *special* intersection of two interval graphs.
- G is a *special* segment graph.

Combinatorial Optimization Problems:

- Weighted Independent set can be solved in polytime on MPT graphs.
- Colouring is NP-complete for MPT graphs.
- 2-Approximation of Clique Cover.

Outline

1. Background and Motivation
2. Properties and Characterizations of MPT graphs
3. Combinatorial Optimization Problems

Linear Ls

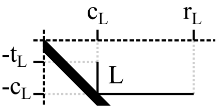
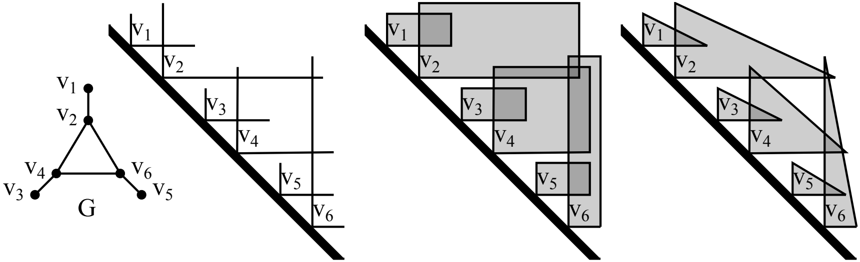


Figure : Anatomy of an L-shape in a linear L-system.

Notice that linear L = linear rectangle = linear right-triangle.



MPT = linear L

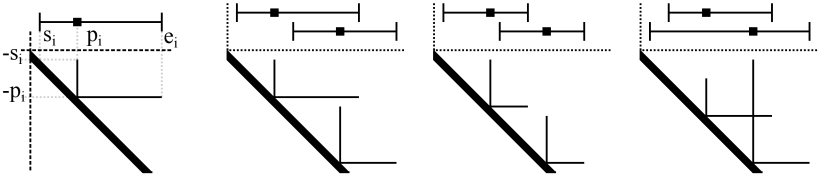
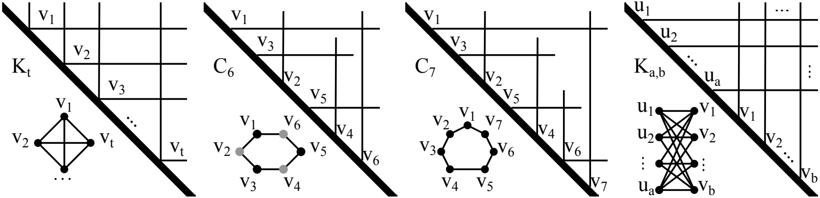


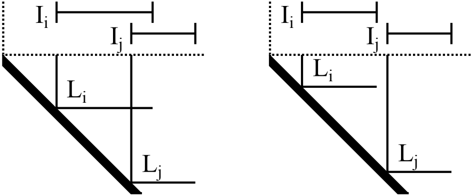
Figure : Illustrating the equivalence between MPT representations and linear L-systems. From left-to-right: the L-shape corresponding to a pointed-interval, two examples of non-adjacent vertices as pointed-intervals and the corresponding linear Ls, and one example of adjacent vertices as pointed-intervals and the corresponding linear Ls.

Some simple linear L-systems



Note: Outerplanar graphs precisely the **contact** linear L-graphs.

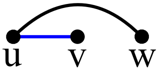
Interval Graphs are ”Anchored” linear Ls



Vertex Orders

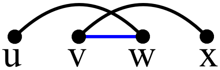
Theorem (Olariu 1991,Ramalingam and Pandu Rangan 1988, Raychaudhuri 1987)

$G = (V, E)$ is an interval graphs iff V can be ordered by $<$ so that for every $u < v < w$, if $uw \in E$, then $uv \in E$.



Theorem

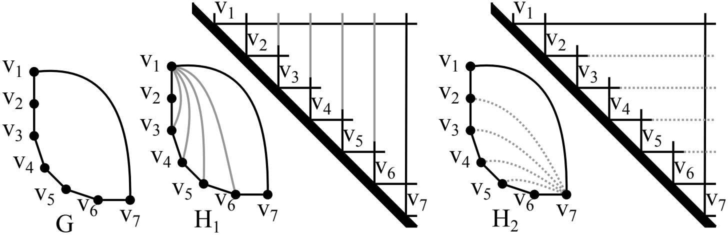
$G = (V, E)$ is an MPT graph iff V can be ordered by $<$ so that for every $u < v < w < x$, if $uw, vx \in E$, then $vw \in E$



MPT graphs as the intersection of two Interval Graphs

Theorem

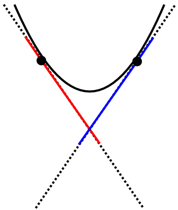
There are two interval graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ such that $E = E_1 \cap E_2$ and the vertices of G can be ordered by $<$ so that for every $u < v < w$ if $uw \in E_1$ then $uv \in E_1$ and if $uw \in E_2$ then $wv \in E_2$.



MPT graph as Segment Graphs

Theorem

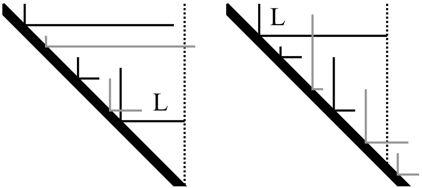
$G = (V, E)$ is an MPT graph iff each vertex can be represented by a line segment tangent to a parabola.



Outline

Weighted Independent Set

idea: right-dominant Ls.



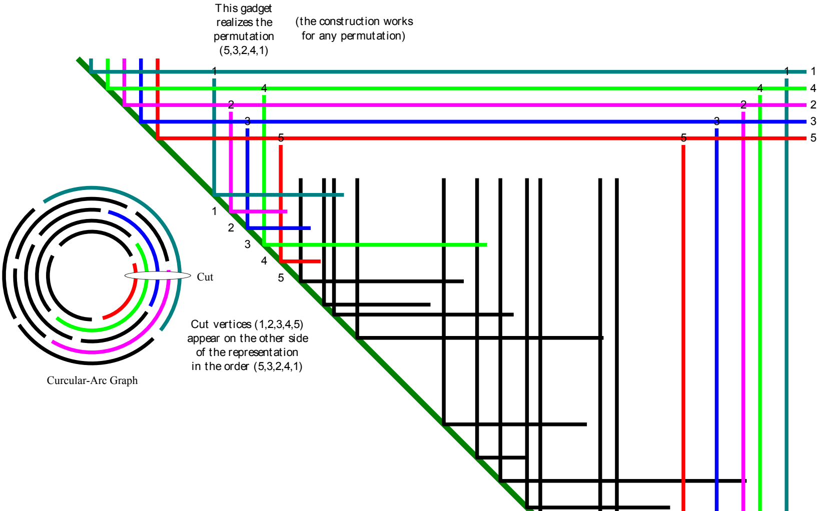
For $a, b \in \{1, ..., n\}$ and $a \le b$, let $\mathcal{L}_{a,b}$ denote the subset of $\{L_a, ..., L_b\}$ which :

- occurs strictly to the left of the line $x = \min\{r_{a-1}, r_{b+1}\}$; and
- includes no neighbours of v_{a-1} (i.e., occurs strictly below the line $y = a - 1$); and
- includes no neighbours of v_{b+1}.

Optimal solution with L_i right-dominant is: $\text{Opt}(\mathcal{L}_{1,i-1}) \cup \{L_i\} \cup \text{Opt}(\mathcal{L}_{i+1,n})$. So, the table we need has $O(n^2)$ entries each of which takes $O(n)$ to compute; i.e., $O(n^3)$ total.

Colouring is NP-complete

From the Hardness of coloring circular arc graphs. [Garey, Johnson, Miller, Papadimitriou; 1980].



2-Approx For Clique Cover

Clique Cover Problem: Partition the graph into a minimum number of cliques.

Algorithm:

- Choose a greedy independent set following an MPT-order.
- Build part of the clique cover from this independent set.
- Remove this partial clique cover. The remainder is an interval graph.
- Construct a clique cover of the remaining interval graph.

Open Problems

- Recognition of MPT graphs.
- k-colouring.
- Other combinatorial optimization problems.

Thank you for your attention!