Finding compatible circuits in eulerian digraphs

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Joint Work with Stephen Hartke

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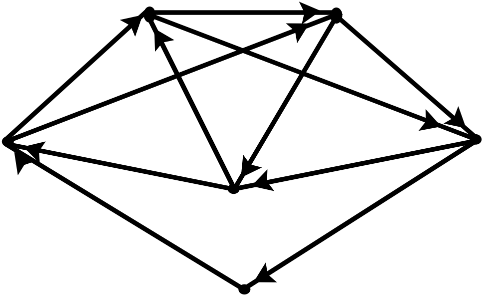
Eulerian digraphs

Def. An *eulerian digraph* G is a digraph that contains a closed walk that visits each edge exactly once.

Thm. A digraph G is eulerian if and only if $\deg^-(v) = \deg^+(v)$ for all vertices v and G is strongly (weakly) connected.



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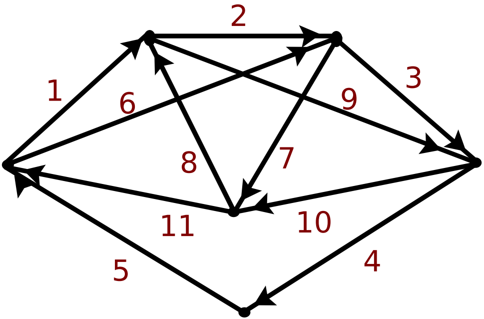
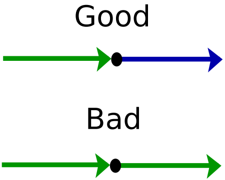
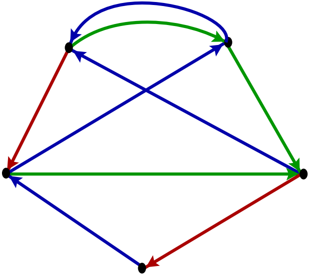


Compatible circuits

Def. A *colored eulerian digraph* G is an eulerian digraph with a fixed edge coloring (not necessarily proper).

A *compatible circuit* is an eulerian circuit of G such that no two consecutive edges in the tour have the same color (i.e. no monochromatic transitions).





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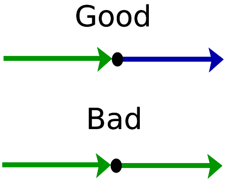
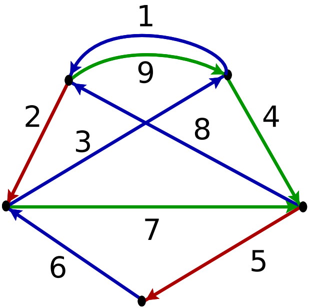
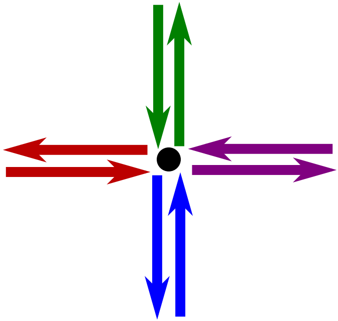




Applications

Eulerian digraphs are applied to routing problems such as garbage collecting, mail carriers, etc.

Restrictions on routes such as no U-turns can be modeled by compatible circuits.

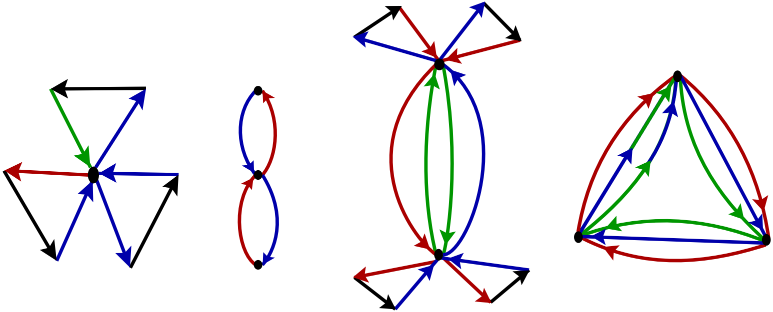


Other applications: universal cycles of permutations, etc.

Examples

Big Question: When does an colored eulerian digraph have a compatible circuit?

Not all graphs have compatible circuits.



Simple necessary condition

Let $\gamma(v)$ be the size of the largest color class incident to v.

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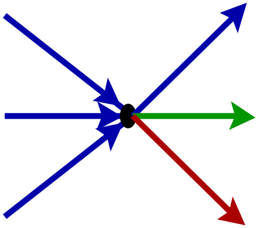

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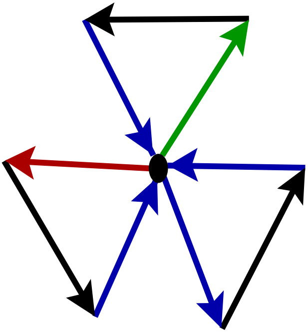

Undirected eulerian graphs

Thm. [Kotzig 1968] If G is a colored eulerian *undirected graph* and $\gamma(v) \leq \deg(v)/2$ for all vertices v, then G has a compatible circuit.

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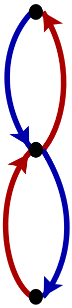
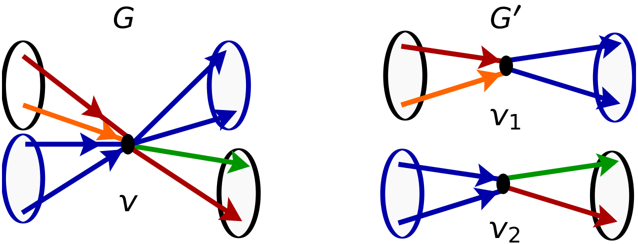


Splitting vertices

We split vertices v where $\gamma(v) = \deg^+(v)$.

The graph G has a compatible circuit if and only if
the graph G' after splitting has a compatible circuit.



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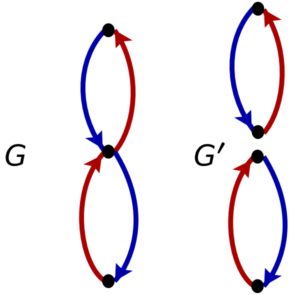

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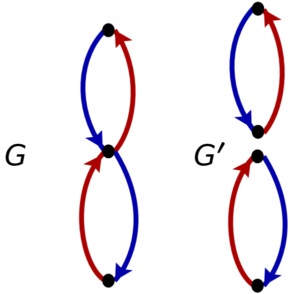
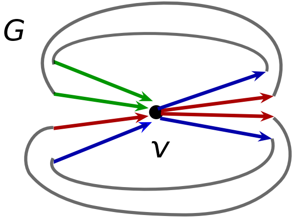


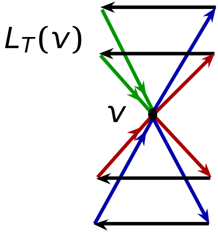
Henceforth, we may assume that $\gamma(v) < \deg^+(v)$ for all v.

Excursions

Def. Let T be an eulerian circuit of G and v a vertex of G. An *excursion* in T is the walk between consecutive visits to v.

The excursion graph $L_T(v)$ tracks the entering and exiting edges of the excursions at v.





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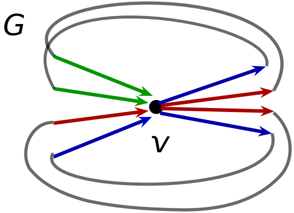
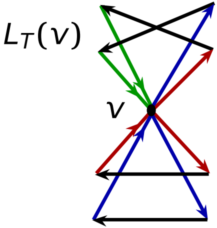
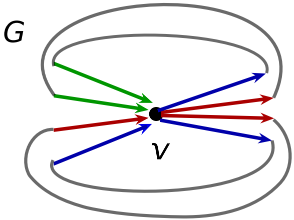
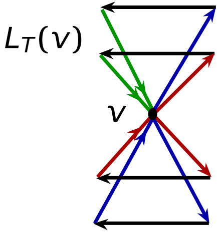




Fixable vertices

We want to remove monochromatic transitions of T at v by rearranging the excursions at v.





Def. Let M be any matching between $E^+(v)$ and $E^-(v)$, and let $L_M(v)$ be the implied excursion graph.

A vertex v is *fixable* if $L_M(v)$ has a compatible circuit for *any* matching M between $E^+(v)$ and $E^-(v)$.

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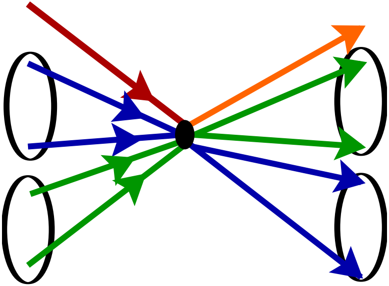
Prop. If every vertex is fixable, then G has a compatible circuit.

Proof.

Pick a (not necessarily compatible) eulerian circuit T of G.
Iteratively fix fixable vertices.
The resulting circuit is compatible. ∎

Prop. A vertex is fixable unless $\gamma(v) = \deg^+(v) - 1$ and there are 2 color classes of size $\gamma(v)$ with both in and out edges, and the other two edges are one incoming and one outgoing.



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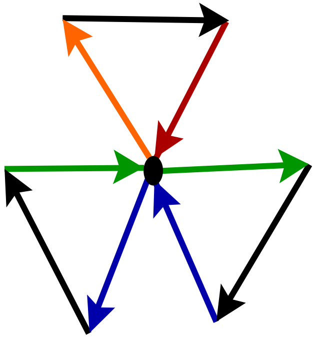
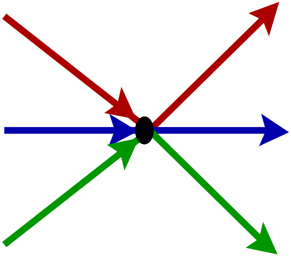
Ex. The excursion graph $L_M(v)$ has no compatible circuit.



Nonfixable vertices

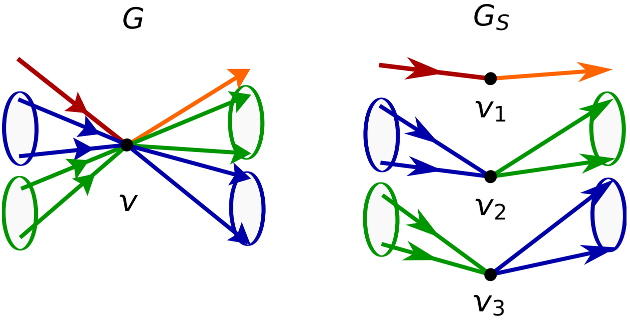
Let S be the set of vertices that are not fixable.
Let S_3 be the subset of S with vertices of outdegree three.



We will consider colored eulerian digraphs
with no nonfixable vertices of outdegree three.

Splitting nonfixable vertices

We form a new graph G_S by splitting each of the nonfixable vertices into three new vertices.



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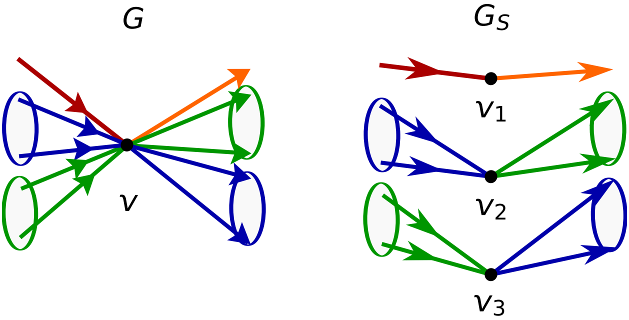
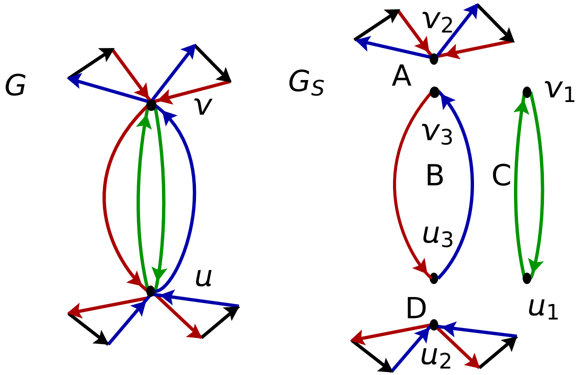


A compatible circuit through v can insert v_1 into v_2 or v_3, but v_2 and v_3 can not be combined.

Can we glue vertices so that the whole graph is connected?

Component graph



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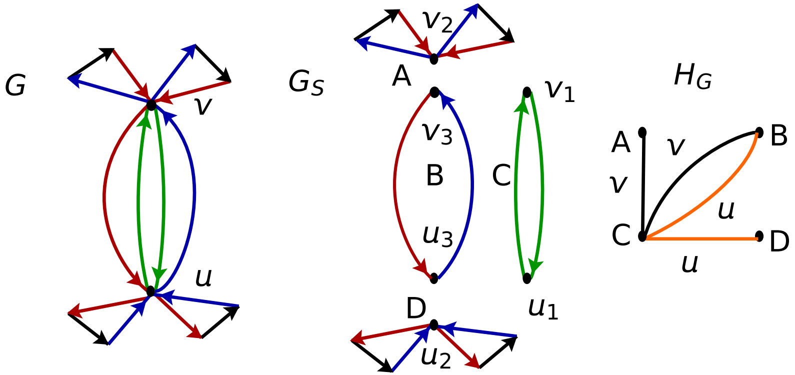


The *component graph H_G* has components of G_S as vertices.
For each $v \in S$, put an edge in H_G between $D_1 \ni v_1$ and $D_2 \ni v_2$ and an edge between $D_1 \ni v_1$ and $D_3 \ni v_3$.

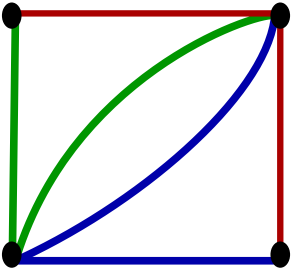
Rainbow spanning trees

Prob. The edge set of H_G is the disjoint union of 2-trails.

Does there exist a subset E' of the edges such that

1. E' contains at most one edge from each 2-trail, and
2. the spanning subgraph with edge set E' is connected?

If so, then H_G contains a *rainbow spanning tree*.



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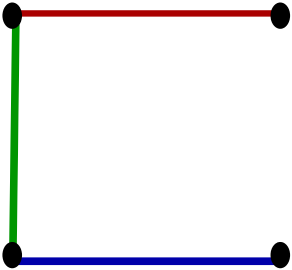

Characterization

Thm. Let G be a colored eulerian digraph
with no nonfixable vertices of outdegree three.

Then G has a compatible circuit if and only if
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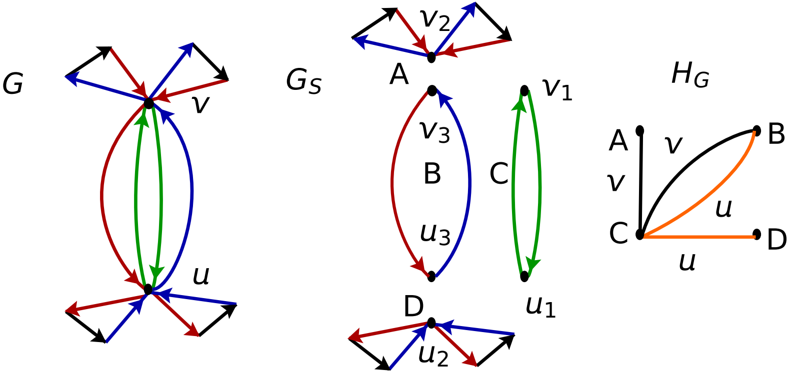
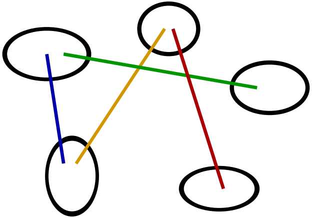


Rainbow spanning trees

Prop. [Broersma and Li 1997; Schrijver 2003; Suzuki 2006]
A multigraph H has a *rainbow spanning tree* if and only if for any partition π of $V(H)$,

$$(\#\text{colors between the parts}) \geq (\#\text{parts in } \pi) - 1.$$



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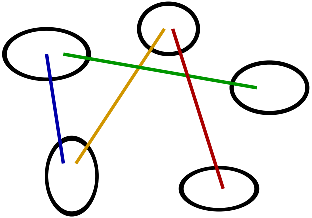
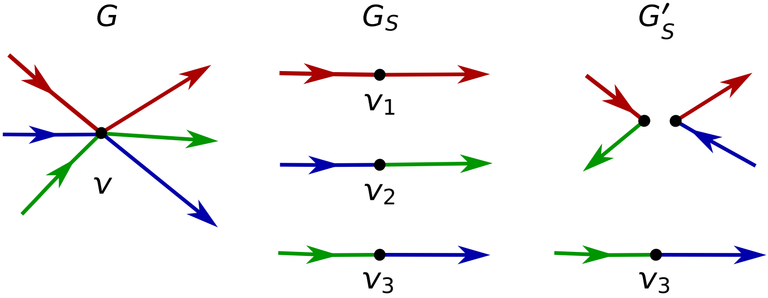


One proof uses the Matroid Intersection Theorem.

There is a polynomial-time algorithm to determine
if a multigraph H contains a rainbow spanning tree.

Nonfixable vertices of outdegree 3

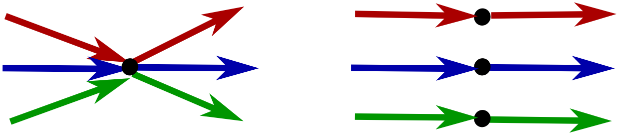
The difficulty with nonfixable vertices of outdegree 3.



A different Component Graph

Let G be an eulerian digraph where all the vertices are nonfixable vertices of outdegree 3.

We construct a component graph by splitting each vertex in the following way.



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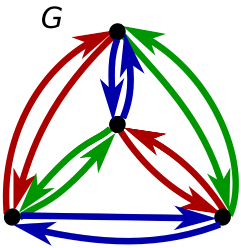
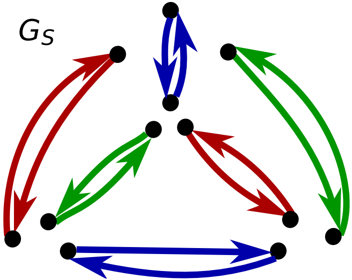
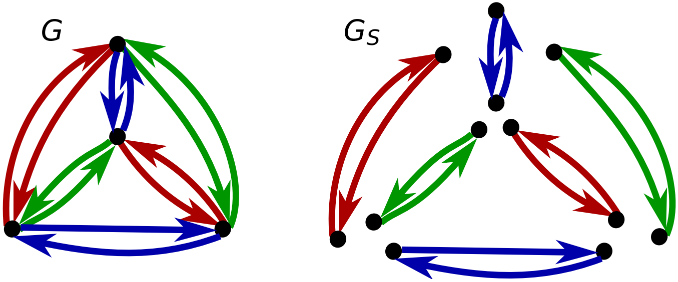


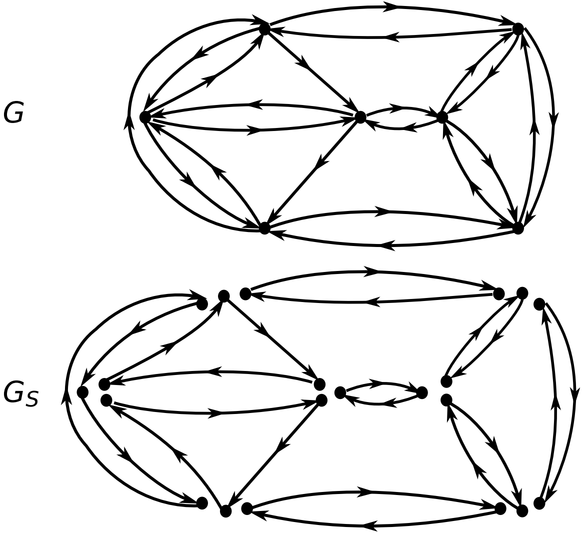


Prop. If G_S has an even number of components then G does not have a compatible circuit.



Prop. This implies that at least half the edge-colorings of G fail, where all vertices of G have outdegree 3.

Let G be a planar digraph where each face is a directed cycle.



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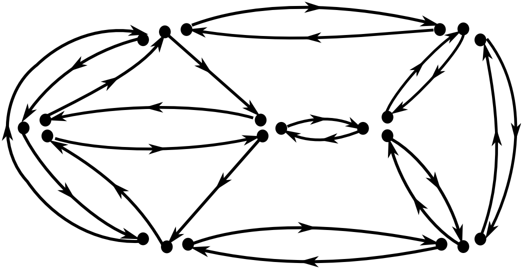
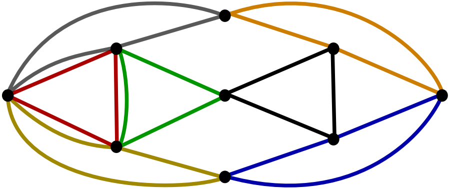
G_S



H_G



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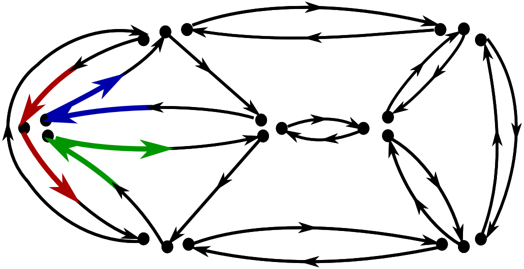

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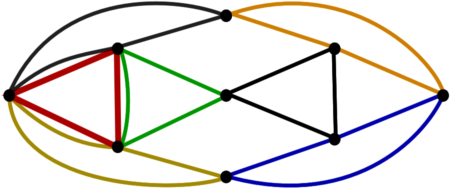

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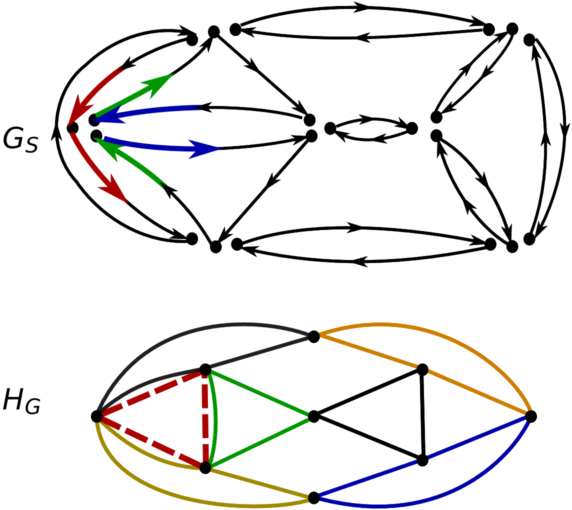
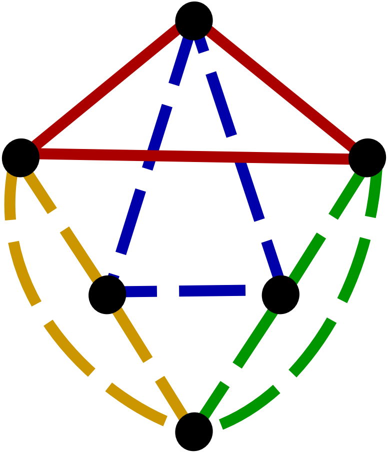


A *spanning cactus* is a spanning subgraph of the component graph such that it has one edge of each dashed triangle and no cycles besides the 3-cycles from the solid triangles.



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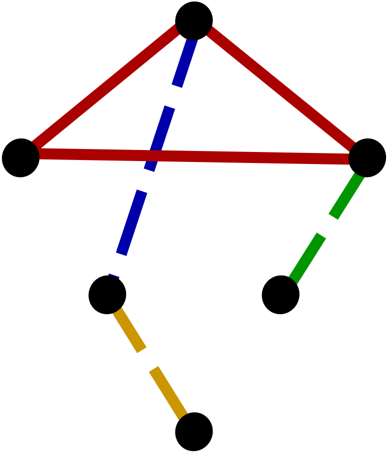
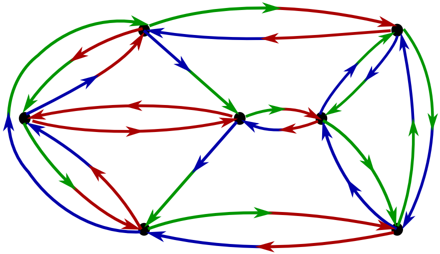


Thm. If G is a planar digraphs where each face is a cycle, then G has a compatible circuit if and only if the component graph has a spanning cactus.

G



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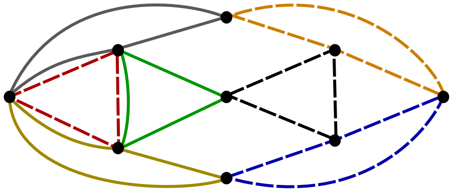

Questions

- Nonfixable vertices of outdegree three?
- Can you find a spanning cactus in H_G in polynomial time?
- Edge-colored Chinese Postman Problem: For a noneulerian graph, minimize both total length of a walk and the number of monochromatic transitions.
- Can you characterize other generalizations or variations of this problem?

Thank You!