Unrolling residues to avoid progressions

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Coloring $1, \ldots, 28$

Take the numbers 1 through 28 and place them in two groups (i.e., color with two colors Red and Blue). And consider the number of monochromatic triples of equally spaced terms.

- How to *maximize*? Easy. We color everything red. 182 such triples.
- How many should we expect at *random*? Easy. Each triple has probability of 1/4 of being monochromatic so at random expect 45.5.
- How to *minimize*? Hard. But thankfully 28 is small!

RRRBBRRRBBBBBBRRRRRRBBBRRBBB

Coloring $1, \ldots, n$

Given that

- we are coloring $1, \ldots, n$
- using r different colors
- trying to avoid monochromatic k-APs

then what is the best we can do?

How few can we get?
What kind of pattern achieves this?

k-APs

A k-term arithmetic progression are k equally spaced integers, i.e., $a, a+d, \ldots, a+(k-1)d$.

Must be many

Theorem (van der Waerden)

For any number r of colors and k of length of arithmetic progressions there is a threshold N so that if $n \geq N$ then *any* coloring of $1, 2, \ldots, n$ using r colors must have a monochromatic arithmetic progression of length k.

Theorem (Frankl-Graham-Rödl)

For fixed r and k, there is $c > 0$ so that the number of monochromatic k-APs in any r-coloring of $1, 2, \ldots, n$ is at least $cn^2 + o(n^2)$.

How about random?

Observation

The number of k-APs in $1, 2, \ldots, n$ is

$$\frac{(n-a)(n-k+1+a)}{2(k-1)} = \frac{n^2}{2(k-1)} + O(n),$$

where $n = (k-1)\ell + a$ and $0 \leq a < k-1$.

Lemma

There is a coloring of $1, 2, \ldots, n$ with r colors which has at most $\frac{1}{2(k-1)r^{k-1}} n^2 + O(n)$ monochromatic k-APs.

Proof: Color randomly. $\square$

Expanding a coloring

Start with a good small coloring and expand it, i.e.,

RRRRRRBBBBRRRRRRBBBBBBBBBBBBRRRRRRRRRRRRBBBBBBRRRRBBBBBB

For n large this gives $\frac{3}{56}n^2 + O(n)$ monochromatic 3-APs, or 21.42% (random is 25%).

Theorem (Parrilo-Robertson-Saracino; Butler-Costello-Graham)

Expanding the following coloring gives $\frac{117}{2192}n^2 + O(n)$ monochromatic 3-APs, or 21.35%:

$$\underbrace{R\cdots R}_{28}\underbrace{B\cdots B}_{6}\underbrace{R\cdots R}_{28}\underbrace{B\cdots B}_{37}\underbrace{R\cdots R}_{59}\underbrace{B\cdots B}_{116}\underbrace{R\cdots R}_{116}\underbrace{B\cdots B}_{59}\underbrace{R\cdots R}_{37}\underbrace{B\cdots B}_{28}\underbrace{R\cdots R}_{6}\underbrace{B\cdots B}_{28}$$

Unrolling a coloring

Start with a good small coloring and repeat, i.e.,

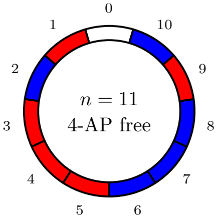
RRRBBRRRBBBBBBRRRRRRBBBRRBBBRRRBBRRRBBBBBBRRRRRRBBBRRBBB

For n large this gives $\frac{1}{16}n^2 + O(n)$ monochromatic 3-APs, the same as random!

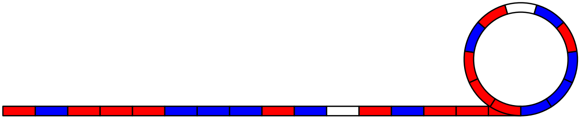
Repeating *any* pattern will *not* beat the random bound for 3-APs…but for k-APs with $k \geq 4$ the story is very different!

Good coloring of $\mathbb{Z}_{11}$

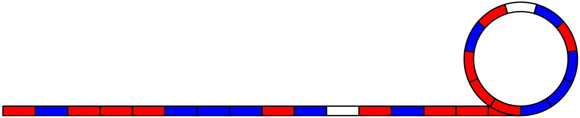
Lu and Peng found the following good coloring of $\mathbb{Z}_{11}$:



Then unrolled it to get a coloring of the integers:



Good coloring of $\mathbb{Z}_{11}$



$$\ell = \sum_i b_i \cdot 11^i \qquad \text{where } 0 \le b_i \le 10.$$

Let j be the smallest index so that $b_j \neq 0$,

$$\text{color } \ell \quad \begin{cases} \text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10. \end{cases}$$

This coloring has $\frac{1}{72}n^2 + O(n)$ monochromatic 4-APs. (Far superior to the best coloring found by expanding blocks.)

Theorem

If there is a coloring of $\mathbb{Z}_m$ with r colors which have no monochromatic k-APs and 0 can be colored arbitrarily, then there is an r-coloring of $1, 2, \ldots, n$ which has $\frac{1}{2(m+1)(k-1)}n^2 + O(n)$ monochromatic k-APs.

Proof: After unrolling we have $m-1$ monochromatic arithmetic progressions of length n/m. The last residue we recursively color.

Let $F(n)$ be the number of monochromatic k-APs then (ignoring lower order terms)

$$F(n) = F\left(\frac{n}{m}\right) + \frac{m-1}{2(k-1)}\left(\frac{n}{m}\right)^2 = \frac{(m-1)n^2}{2(k-1)}\sum_{i\geq 1}\left(\frac{1}{m^2}\right)^i$$

$$= \frac{(m-1)n^2}{2(k-1)} \cdot \frac{1}{m^2-1} = \frac{1}{2(m+1)(k-1)}n^2. \qquad \square$$

What do we unroll?

$$\text{color } \ell \quad \begin{cases} \text{red} & \text{if } b_j = 1, 3, 4, 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2, 6, 7, 8, \text{ or } 10. \end{cases}$$

Note that $\{1, 3, 4, 5, 9\}$ are the quadratic residues of $\mathbb{Z}_{11}$!

Quadratic residues are good for 2 colors:

- Easy to see if k-AP free (i.e., only have to check for longest run of residues).
- Longest run cannot be too big: $O\left(p^{1/4}(\log p)^{3/2}\right)$ (Burgess)

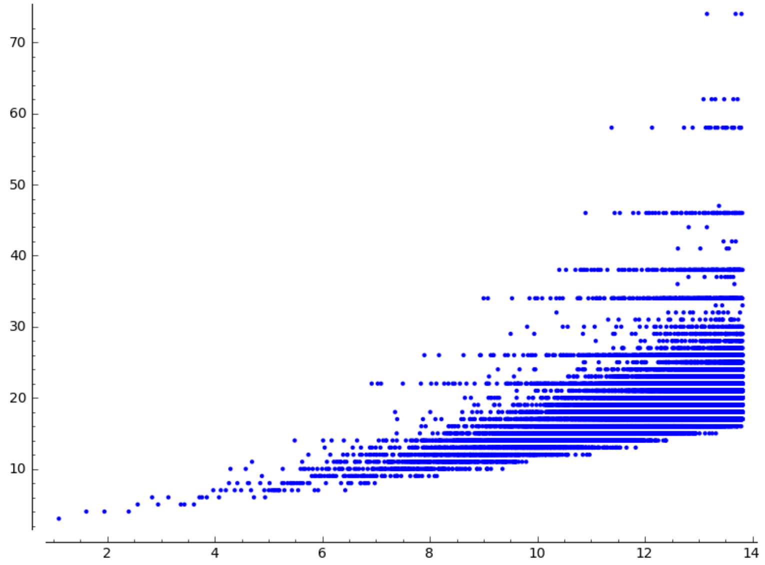
More generally

For more colors use higher order residues, i.e., $\{x^r \mid x \in \mathbb{Z}_p, x \neq 0\}$.

Theorem

For $i = 1, 2$, let $\mathcal{C}_i$ be a coloring of $\mathbb{Z}_{m_i}$ using r_i colors where 0 can be colored arbitrarily and containing no nontrivial k-APs. Then there exists a coloring $\mathcal{C}$ of $\mathbb{Z}_{m_1 m_2}$ using $r_1 r_2$ colors where 0 can be colored arbitrarily and containing no nontrivial k-APs.

Large p with small runs

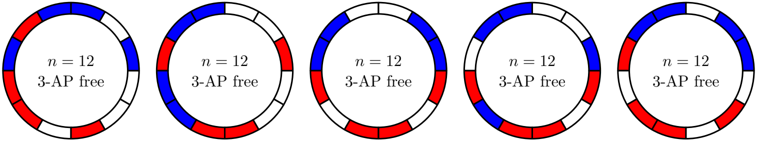


Best known results

m	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7
3			37		103	
4	11	97	349	751	3259	1933
5	37	241	2609	6011	14173	30493
6	139	1777	139^2	49391	$139 \cdot 1777$	317969
7	617	7309	617^2	230281	$617 \cdot 7309$	
8	1069	34057	1069^2			
9	3389	116593				
10	11497	463747				
11	17863					
12	58013					
13	136859					

What about $k = 3$, $r = 3$?

Cannot use residues since $-1, 0, 1$ are all cubic residues.
But we don't have to use residues to do unrolling.



This gives $\frac{1}{48}n^2 + O(n)$ monochromatic 3-APs, so 8.33% of the colorings will be monochromatic, whereas in a random coloring we would expect 11.11% of the 3-APs to be monochromatic.

Open problems/Conclusion

- We have done constructions and found "upper bounds" for the best colorings. What about "lower bounds".
- For $k = 3$ and $r = 2$ show $\geq \frac{117}{2192}n^2 + O(n)$.
- Show that we can always beat random.

- Is unrolling almost always best?
- Are residues almost always the best thing to unroll?
- What about avoiding non-APs?
- Thank you.