Broadcast domination and its dual multipackings

CanaDAM 2013

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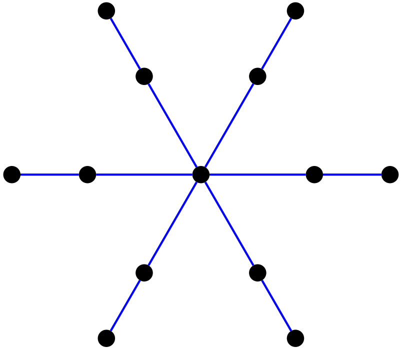
Domination and packing



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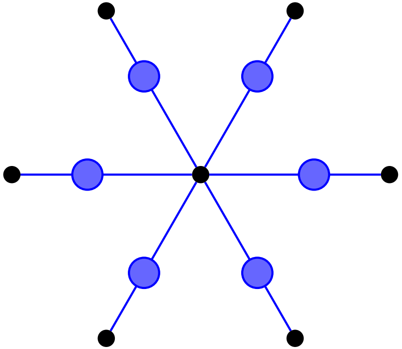

Dominating set S

$\gamma = 6$

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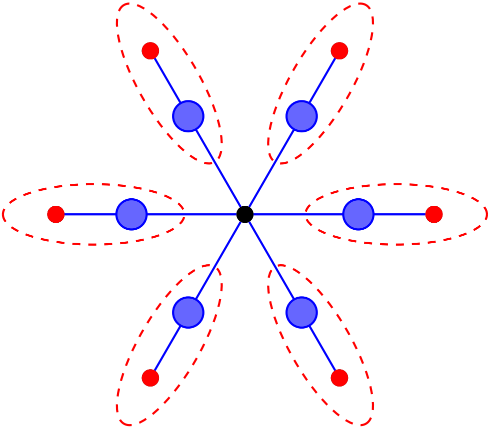

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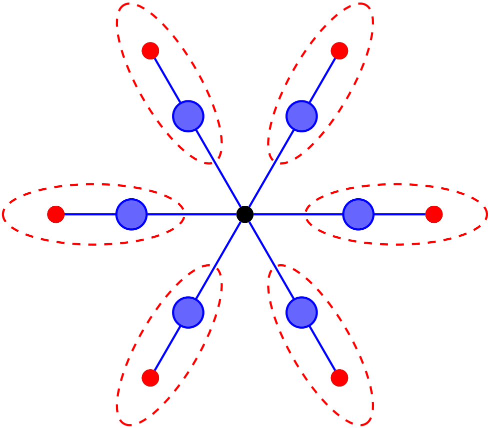

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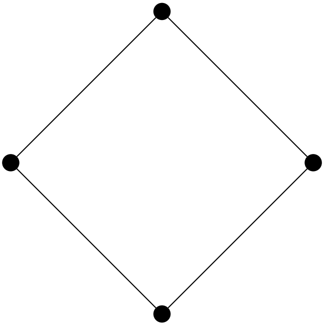
Packing of disjoint nbhds

Meir and Moon, 1975, $P_2(T) = \gamma$ for a tree.
For all v, $N[v]$ covers at most 1 red vertex.

Cycles



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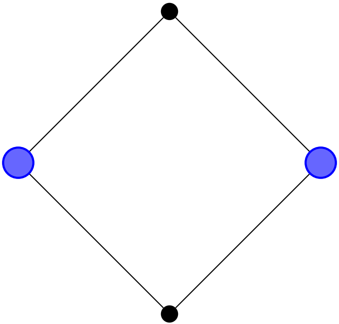

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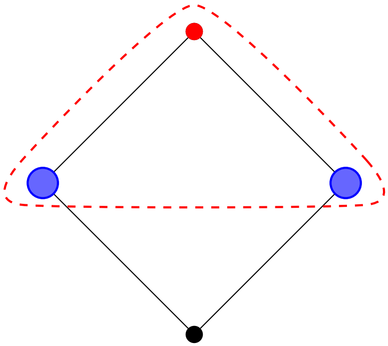

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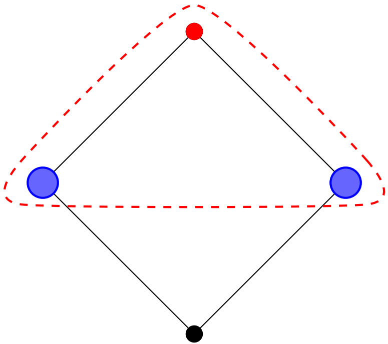
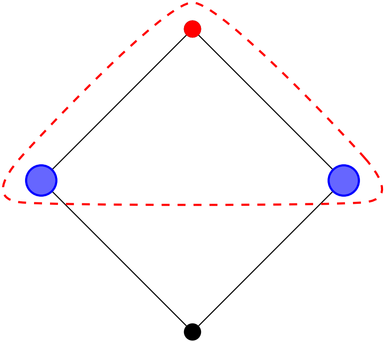


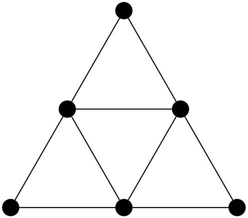
- $P_2(C_4) = 1 < \gamma = 2.$

Cycles



- $P_2(C_4) = 1 < \gamma = 2$.
- Equality for $n \equiv 0 \pmod 3$

Between trees and cycles?



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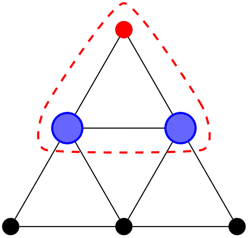

- **Chordal graphs** do not have equality in general. $\gamma(S_3) = 2$ but $P_2(S_3) = 1$. (No dominating vertex and $\mathrm{rad}(S_3) = 2$.)

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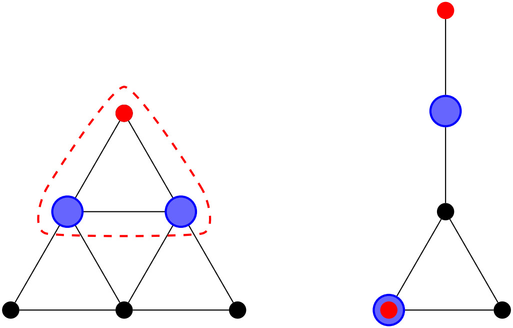

- **Chordal graphs** do not have equality in general. $\gamma(S_3) = 2$ but $P_2(S_3) = 1$. (No dominating vertex and $\mathrm{rad}(S_3) = 2$.)
- For **strongly chordal graphs** $P_2(G) = \gamma(G)$. [Farber, 1984]

Coverings and packings

- A dominating set may be viewed as a cover of the graph by closed neighbourhoods. (Every vertex belongs to at least one closed neighbourhood.)

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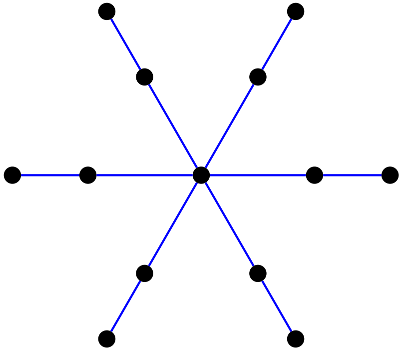
- **Strongly chordal graphs** $P_2(G) = \gamma(G)$. Domination linear time solvable.

Broadcasting



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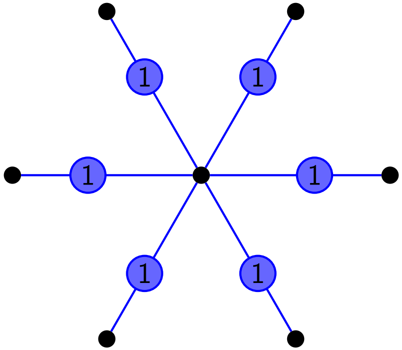

- Cover with 6 closed neighbourhoods (balls of radius 1).

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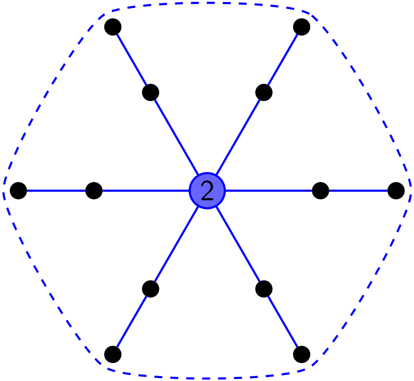

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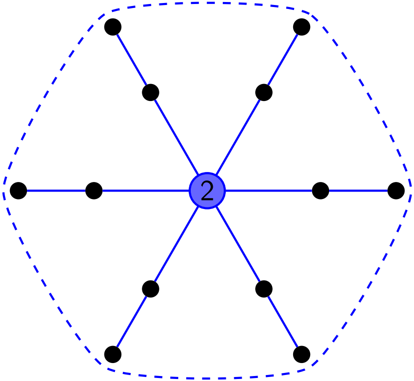

- Cover with 6 closed neighbourhoods (balls of radius 1).
- Cover with a single two-neighbourhood (ball of radius 2).
- Define the **cost of a ball to be its radius**. The first cover costs 6, the second cost 2.

Broadcast Domination

Introduced by Erwin, (2001, 2004)

Definition

Let $G = (V, E)$ be a graph.

- A *broadcast* is $f : V(G) \to \mathbb{N}$ such that $\forall v, f(v) \leq ecc(v)$;
- A *dominating broadcast* has the additional property that for all u there is v such that $f(v) > 0$ and $d(u, v) \leq f(v)$;
- The *cost* of f is $\sigma(f) = \sum_{v \in V} f(v) = \sum_{f(v)>0} f(v)$;
- The *broadcast number* $\gamma_b = \min_f \sigma(f)$ where f is a dominating broadcast.

Bounds

- $\gamma_b(G) \leq \min\{\mathrm{rad}(G), \gamma\}$ [Erwin, 2004]

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- $\gamma_b(G) \geq \left\lceil \frac{\mathrm{diam}(G)+1}{3} \right\rceil$ [Erwin, 2004]
- The subdivided star shows $\gamma - \gamma_b$ can be arbitrarily large.

Algorithms

- Heggernes and Lokshtanov [2006] give a poly-time algorithm to find γ_b. ($O(n^6)$)

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- Dunbar et. al. [2006] Can always find an efficient broadcast domination.
- **Question:** What is the packing lower bound in this context?

IP/LP Formulation

Restating the problem

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- Let $N_k[v]$ denote the ball of radius k around v.)
- The problem of finding a broadcast f is to select a family $\mathcal{B}$ of balls that covers the graph.
- The cost of $\mathcal{B}$ is

$$\sigma(f) = \sum_{N_k[v] \in \mathcal{B}} k$$

IP/LP Formulation

Formally

- Let c, x be vectors indexed by (v, k) $(v \in V,\ 1 \leq k \leq ecc(v))$
- Let $x_{v,k} \in \{0, 1\}$ and $c_{v,k} = k$.
- Let A be the vertex-ball incidence matrix: vertices $\times$ balls.

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$$\begin{aligned} & \min c \cdot x \\ s.t. \quad & Ax \geq \mathbf{1} \\ & x_{v,k} \in \{0, 1\} \end{aligned}$$

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- Multipacking:

$$\begin{aligned} & \max y \cdot \mathbf{1} \\ s.t. \quad & A^t y \leq c \\ & y_v \in \{0, 1\} \end{aligned}$$

Multipackings, the dual

Definition

A set of vertices $S \subseteq V$ is a multipacking if $|S \cap N_k[v]| \leq k$ for v, k. The multipacking number $mp(G)$ is the size of a largest multipacking in G.

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Proposition

$mp(G) \leq \gamma_b(G)$

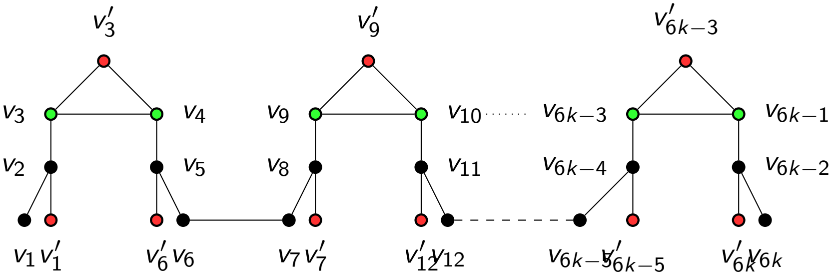
Proof.

$$|S| = \left| S \bigcap \left(\bigcup_{N_k[v] \in \mathcal{B}} N_k[v] \right) \right| = \left| \bigcup_{N_k[v] \in \mathcal{B}} S \cap N_k[v] \right|$$

$$\leq \sum_{N_k[v] \in \mathcal{B}} |S \cap N_k[v]| \leq \sum_{N_k[v] \in \mathcal{B}} k = \sigma(f)$$

□

Laura's example



- $\mathrm{ir}(2k)$ and $\gamma_b = 3k$ (Green vertices are an ir-set. Top vertex of each house broadcast with radius 3.)

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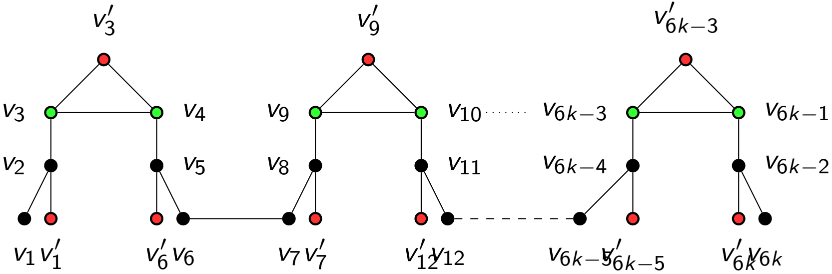

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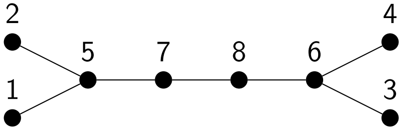
- $\mathrm{mp} = 3k$

Farber's algorithm



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1	1	0	0	0	1	0	0	1	0	0	1	1
2	0	1	0	0	1	0	0	1	0	0	1	1
3	0	0	1	0	0	1	0	0	0	1	1	1
4	0	0	0	1	0	1	0	0	0	1	1	1
5	1	1	0	0	1	0	1	1	0	1	1	1
6	0	0	1	1	0	1	0	1	1	1	1	1
7	0	0	0	0	1	0	1	1	1	1	1	1
8	0	0	0	0	0	1	1	1	1	1	1	1

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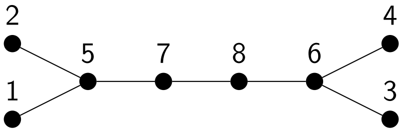

y_i		1_1	2_1	3_1	4_1	5_1	6_1	7_1	7_2	8_1	8_2	7_3	8_3
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0	2	0	1	0	0	1	0	0	1	0	0	1	1
0	3	0	0	1	0	0	1	0	0	0	1	1	1
0	4	0	0	0	1	0	1	0	0	0	1	1	1
0	5	1	1	0	0	1	0	1	1	0	1	1	1
0	6	0	0	1	1	0	1	0	1	1	1	1	1
0	7	0	0	0	0	1	0	1	1	1	1	1	1
0	8	0	0	0	0	0	1	1	1	1	1	1	1
Slack		1	1	1	1	1	1	1	2	1	2	3	3

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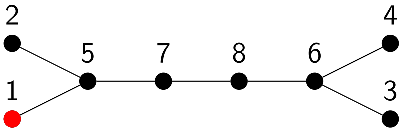

y_i		1_1	2_1	3_1	4_1	5_1	6_1	7_1	7_2	8_1	8_2	7_3	8_3
1	1	1	0	0	0	1	0	0	1	0	0	1	1
0	2	0	1	0	0	1	0	0	1	0	0	1	1
0	3	0	0	1	0	0	1	0	0	0	1	1	1
0	4	0	0	0	1	0	1	0	0	0	1	1	1
0	5	1	1	0	0	1	0	1	1	0	1	1	1
0	6	0	0	1	1	0	1	0	1	1	1	1	1
0	7	0	0	0	0	1	0	1	1	1	1	1	1
0	8	0	0	0	0	0	1	1	1	1	1	1	1
Slack		0	1	1	1	0	1	1	1	1	2	2	2

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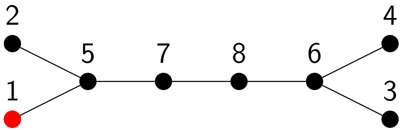

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0	4	0	0	0	1	0	1	0	0	0	1	1	1
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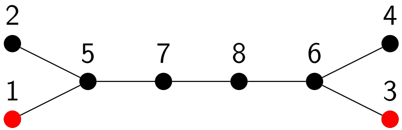

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0	4	0	0	0	1	0	1	0	0	0	1	1	1
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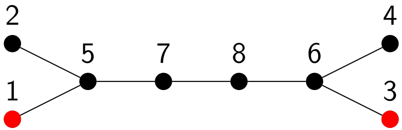

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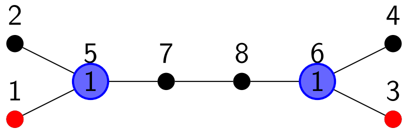

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1	3	0	0	1	0	0	1	0	0	0	1	1	1
0	4	0	0	0	1	0	1	0	0	0	1	1	1
0	5	1	1	0	0	1	0	1	1	0	1	1	1
0	6	0	0	1	1	0	1	0	1	1	1	1	1
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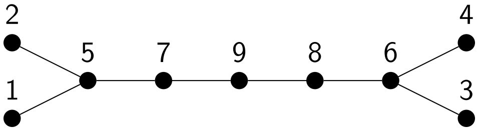

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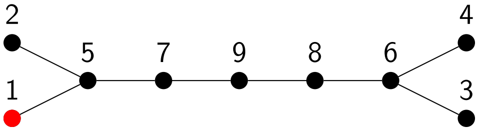
2	0	1	0	0	1	0	0	1	0	0	0	0	1
3	0	0	1	0	0	1	0	0	0	1	0	0	1
4	0	0	0	1	0	1	0	0	0	1	0	0	1
5	1	1	0	0	1	0	1	1	0	0	0	1	1
6	0	0	1	1	0	1	0	0	1	1	0	1	1
7	0	0	0	0	1	0	1	1	0	1	1	1	1
8	0	0	0	0	0	1	0	1	1	1	1	1	1
9	0	0	0	0	0	0	1	1	1	1	1	1	1



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2	0	1	0	0	1	0	0	1	0	0	0	0	1
3	0	0	1	0	0	1	0	0	0	1	0	0	1
4	0	0	0	1	0	1	0	0	0	1	0	0	1
5	1	1	0	0	1	0	1	1	0	0	0	1	1
6	0	0	1	1	0	1	0	0	1	1	0	1	1
7	0	0	0	0	1	0	1	1	0	1	1	1	1
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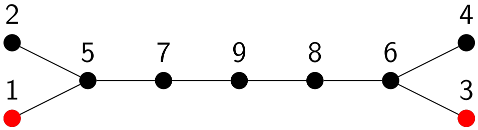

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2	0	1	0	0	1	0	0	1	0	0	0	0	1
3	0	0	1	0	0	1	0	0	0	1	0	0	1
4	0	0	0	1	0	1	0	0	0	1	0	0	1
5	1	1	0	0	1	0	1	1	0	0	0	1	1
6	0	0	1	1	0	1	0	0	1	1	0	1	1
7	0	0	0	0	1	0	1	1	0	1	1	1	1
8	0	0	0	0	0	1	0	1	1	1	1	1	1
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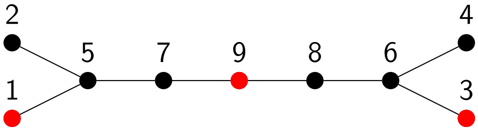

	1_1	2_1	3_1	4_1	5_1	6_1	7_1	7_2	8_1	8_2	9_1	9_2	9_3
1	1	0	0	0	1	0	0	1	0	0	0	0	1
2	0	1	0	0	1	0	0	1	0	0	0	0	1
3	0	0	1	0	0	1	0	0	0	1	0	0	1
4	0	0	0	1	0	1	0	0	0	1	0	0	1
5	1	1	0	0	1	0	1	1	0	0	0	1	1
6	0	0	1	1	0	1	0	0	1	1	0	1	1
7	0	0	0	0	1	0	1	1	0	1	1	1	1
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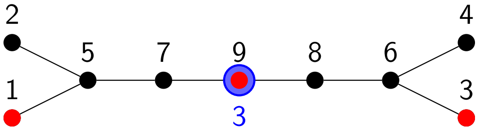


	1_1	2_1	3_1	4_1	5_1	6_1	7_1	7_2	8_1	8_2	9_1	9_2	9_3
1	1	0	0	0	1	0	0	1	0	0	0	0	1
2	0	1	0	0	1	0	0	1	0	0	0	0	1
3	0	0	1	0	0	1	0	0	0	1	0	0	1
4	0	0	0	1	0	1	0	0	0	1	0	0	1
5	1	1	0	0	1	0	1	1	0	0	0	1	1
6	0	0	1	1	0	1	0	0	1	1	0	1	1
7	0	0	0	0	1	0	1	1	0	1	1	1	1
8	0	0	0	0	0	1	0	1	1	1	1	1	1
9	0	0	0	0	0	0	1	1	1	1	1	1	1

Farber's algorithm

- Step 1: Greedy algorithm to construct a feasbile dual solution. (Top to bottom)
- Step 2: Choose balls with zero dual slack (right to left) so that packing vertices belong to at most one ball.
- At this point conditions for *complementary slackness are met*.

Theorem (Farber)

If A is totally balanced, then primal solution is feasible.

- Totally balanced absence of

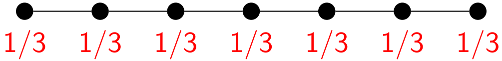
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- Strongly chordal neighbourhood matrix is totally balanced.

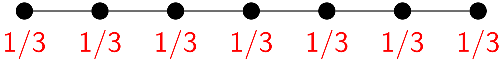
Recap

- Farber's two pass (primal-dual) algorithm.
- Linear time $O(n + m)$.
- Works when A is totally-balanced.
- Can be generalized to γ_b.

Fractional relaxation

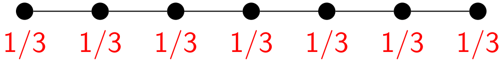


Fractional relaxation



- Assign each vertex weight $wt(v) = 1/3$.
- $wt(N_k[v]) \leq \frac{2k+1}{3} \leq k$
- This is a feasible *fractional* packing.

Fractional relaxation



- Assign each vertex weight $wt(v) = 1/3$.
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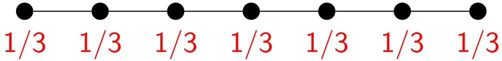
Fractional relaxation



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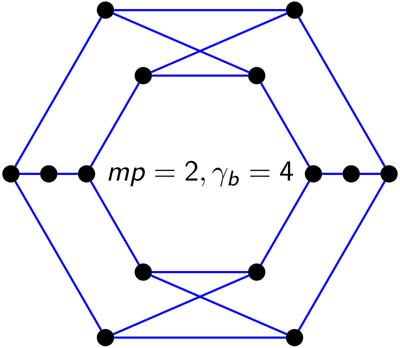
- Easy certificate $\gamma_b(G) \geq \left\lceil \frac{\mathrm{diam}(G)+1}{3} \right\rceil$.

Fractional relaxation

- Assign each vertex weight $wt(v) = 1/3$.
- $wt(N_k[v]) \leq \frac{2k+1}{3} \leq k$
- This is a feasible *fractional* packing.
- Easy certificate $\gamma_b(P_n) \geq \left\lceil \frac{n}{3} \right\rceil$.
- Easy certificate $\gamma_b(G) \geq \left\lceil \frac{\mathrm{diam}(G)+1}{3} \right\rceil$.
- We have several examples where a fractional packing gives an easy lower bound within one of γ_b.

Mind the gap



- Found by Mynhardt and Teshima
- We have generalized this family. $\gamma_b - mp$ grows.

Questions

- For what graphs does $mp = \gamma_b$? (D.R. Fulkerson, A.J. Hoffman and R. Oppenheim, 1975, show the packing and covering LPs have (0,1)-vertices when A is balanced.)
- Farber's algorithm works for totally-balanced A. Linear time for weighted domination. How does the complexity change for broadcast domination?
- Classes of families where the fractional packing number gives a useful lower bound?
- Explore the difference between the polynomial time algorithm for all graphs versus this IP/LP approach.