Packing Tree Factors in (Pseudo) Random Graphs

CanaDAM 2013 - Memorial University

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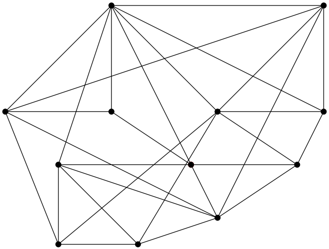
H-factors

Let H be a fixed graph on t vertices.

Definition

An *H-factor* in a graph $G = (V, E)$ (with $|V|$ divisible by t) is a collection of vertex disjoint (not necessarily induced) copies of H which covers V.



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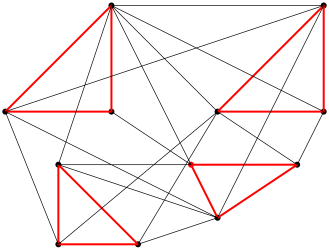

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where H has v vertices, m edges and $d := \frac{m}{v-1}$

- As a corollary, they solve "Shamir's Problem": determine the threshold for the appearance of a perfect matching in $H_{n,p;k}$

Packing Spanning Objects

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- In fact: for all $0 \le p = p(n) \le 1$, $G_{n,p}$ contains $\lfloor \delta/2 \rfloor$ (an optimal number of) edge-disjoint Hamilton cycles whp.(Proved for different ranges of p in many papers)

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- We ask this question for tree factors instead of Hamilton cycles.

Pseudo-Random Graphs

First, lets define our notion of pseudo-randomness:

Definition

Let $G = (V, E)$ be a graph with n vertices. We say G is (ϵ, p)-*regular* if the following 2 conditions hold:

- $d(v) \geq (1-\epsilon)np$ for every vertex v.
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By Chernoff bounds, $G_{n,p}$ satisfies this definition as long as

$$p \gg \frac{1}{\epsilon}\sqrt{\frac{\log n}{n}}$$

1 Background

2 Results

3 Proof Ideas

Result for Pseudo-Random Graphs

Theorem

Let T be a fixed tree with t vertices and let G be an (ϵ, p)-regular graph on n vertices with n a sufficiently large multiple of t. There exists a $C = C(\epsilon)$ such that if $p \geq C\frac{\log^{3/4} n}{n^{1/4}}$ then G contains a collection of edge disjoint T-factors covering all but $2\epsilon^{1/3}$-fraction of the edges of G.

Results for Random Graphs

Theorem

Let T be a fixed tree with t vertices. For any $\epsilon > 0$, there exists $C = C(\epsilon)$ such that if $p \geq C\frac{\log^2 n}{n}$ and n is a multiple of t, then $G_{n,p}$ contains a collection of edge disjoint T-factors covering all but $O(\epsilon)$-fraction of its edges **whp**.

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Theorem

Let T be a fixed tree with t vertices. Then for $\epsilon > 0$, a small number, there exists τ_0 such that for any $\tau \geq \tau_0$ with $t|\tau$, there exists a constant C such that if $p \geq C\frac{\log n}{n}$ and n is a multiple of τ, then $G_{n,p}$ contains a collection of edge disjoint T-factors covering all but $O(\epsilon)$-fraction of its edges **whp**.

1 Background

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Two Useful Lemmas

Frieze and Krivelevich proved the following:

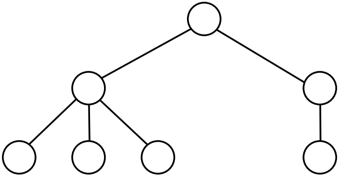
Lemma (Pseudo-Random Bipartite Decomposition)

Any sufficiently large (ϵ, p)-regular bipartite graph with $p \gg 1/\sqrt{n}$ contains a collection of edge-disjoint perfect matchings which covers all but a $O(\epsilon^{1/3})$-fraction of its edges

Lemma (Random Bipartite Decomposition)

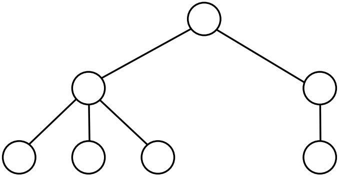
$B_{n,n,p}$ with $p \gg \log n/n$ contains a collection of edge-disjoint perfect matchings which covers all but a $o(1)$-fraction of its edges **whp**.

Proof Idea



Fixed tree T

Proof Idea



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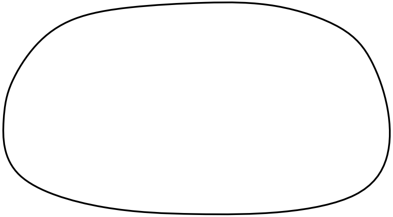
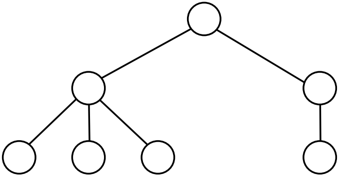


Host Graph G

Proof Idea

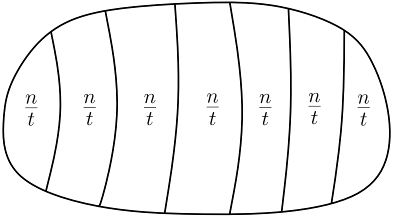
$|T| = t$



Fixed Tree T

Partition σ

$|G| = n$



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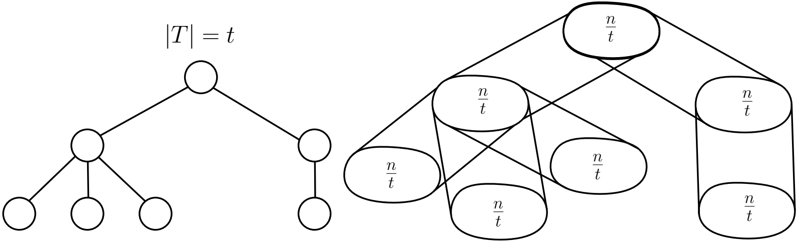

Fixed Tree T

New Graph G_σ

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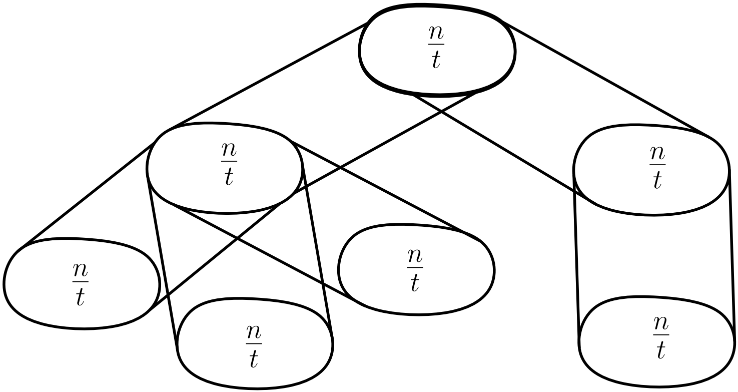

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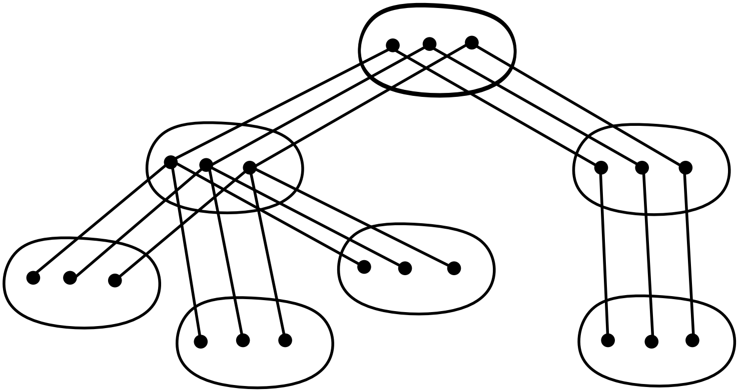

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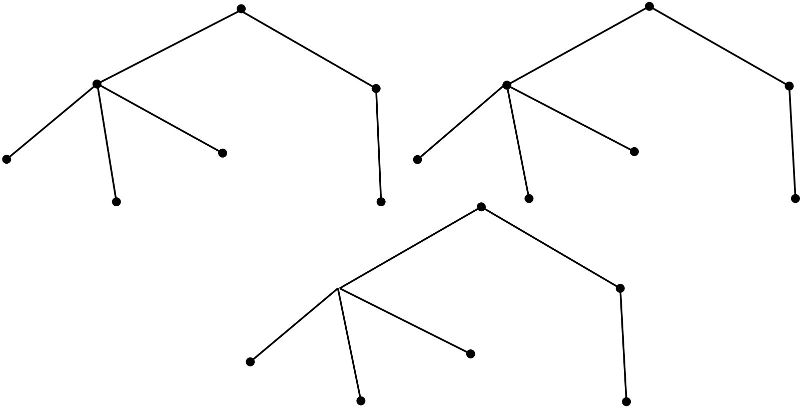
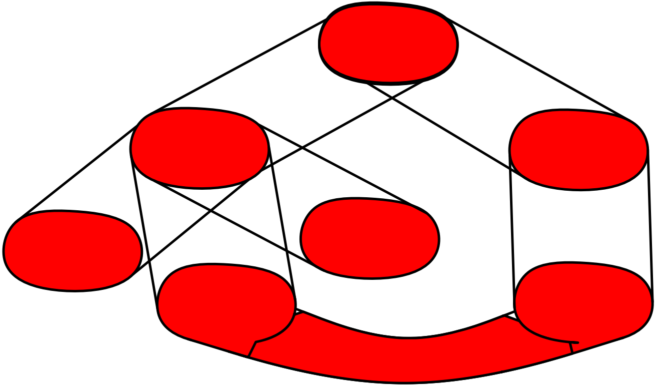


Proof Idea

The problem is that we leave behind edges like those in areas shaded red below:



A Procedure

So we do the following procedure to the input graph G ((ϵ, p)-regular or $G_{n,p}$):

1. Generate $r = K \log n$ independent random partitions of the vertex set of G and let $G_1, \ldots, G_r$ be the graphs corresponding to those partitions.

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3. Let each edge choose uniformly randomly a graph G_i, which contains it.
4. For $i = 1, \ldots, r$, let $\widehat{G}_i \subset G_i$ be the graph of all the edges that chose G_i.

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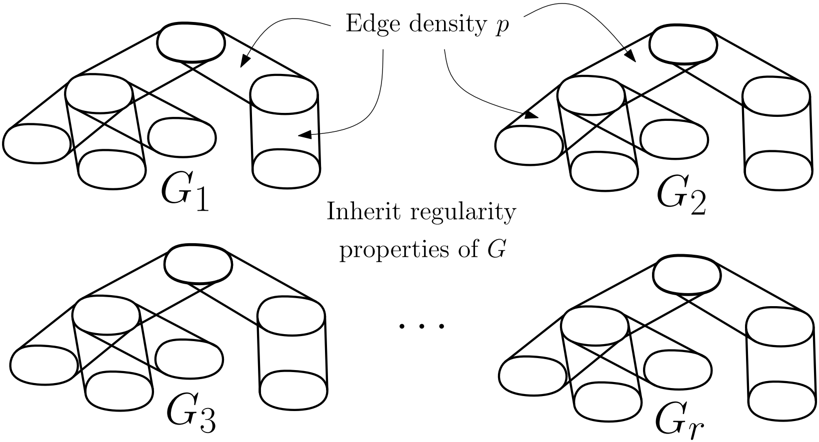
Edge density p
G_1
G_2
Inherit regularity properties of G
G_3
$\dots$
G_r

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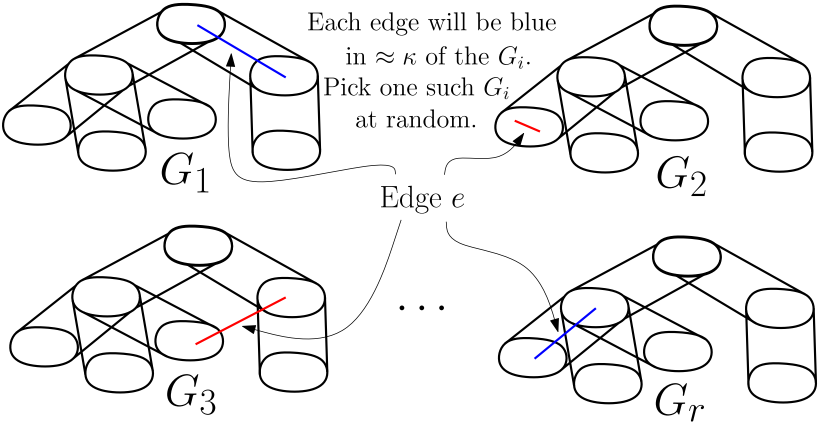
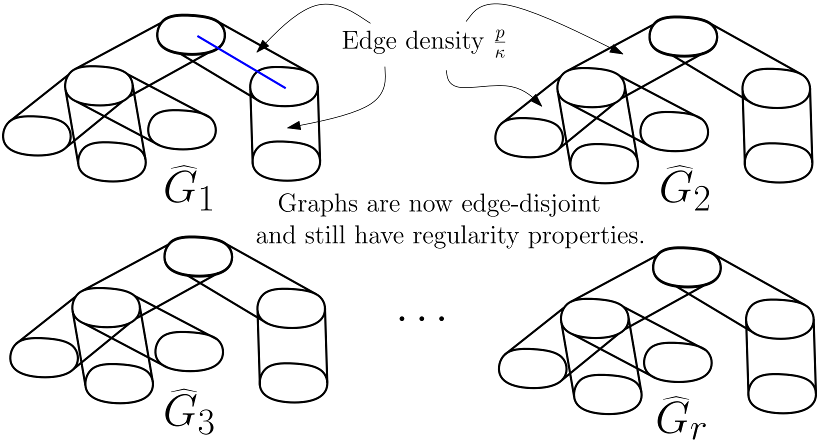

Each edge will be blue
in $\approx \kappa$ of the G_i.
Pick one such G_i
at random.
Edge e
G_1
G_2
G_3
$\cdots$
G_r

Proof Idea


Edge density $\frac{p}{\kappa}$
$\widehat{G}_1$
$\widehat{G}_2$
Graphs are now edge-disjoint
and still have regularity properties.
$\widehat{G}_3$
. . .
$\widehat{G}_r$

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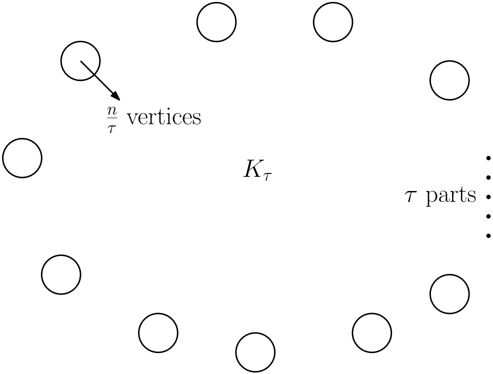
- So we may apply the appropriate perfect matching packing lemmas across the pairs.
- This is why $\log^2 n/n$ is necessary to apply this technique in $G_{n,p}$ case.
- Only edges lost are those left behind by these lemmas, and these are negligible compared to the total number.

Extra Idea Needed for $C \log n/n$.

Split $G_{n,p}$ as follows and pretend its parts are the vertices of K_τ.



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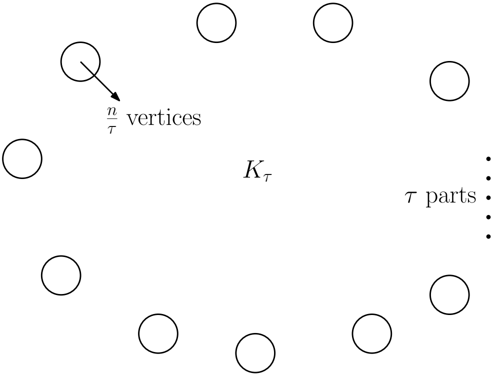
K_τ for τ sufficiently large is $(\epsilon, 1)$-regular!



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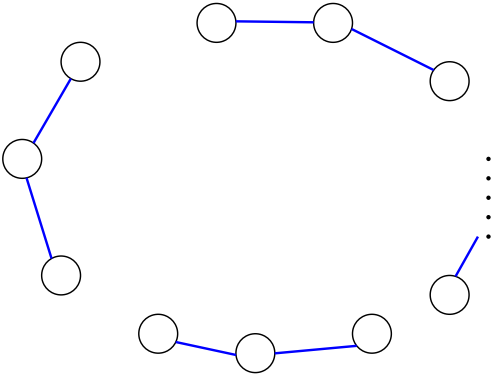

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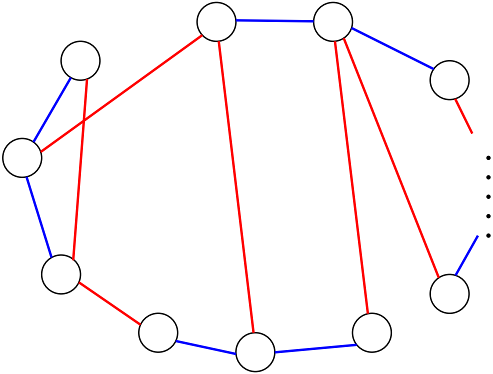

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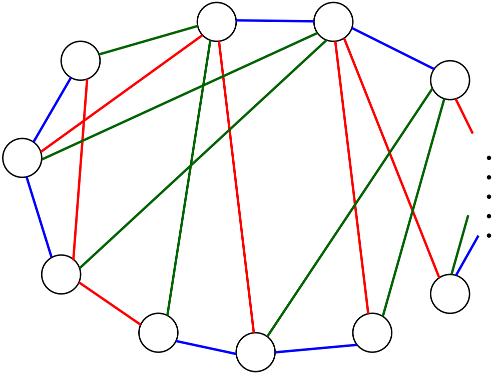

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