# Two Laplacians for the distance matrix of a graph

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# PLAN

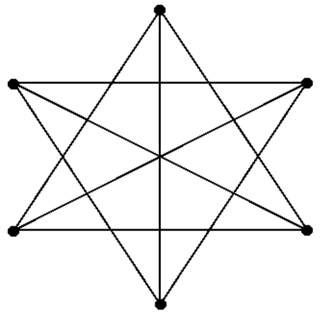
1. Adjacency related matrices
2. Distance Matrix
3. Distance Laplacian matrix
4. Distance signless Laplacian matrix

## Adjacency matrix

- For a graph $G = (V, E)$ on $n$ vertices, the adjacency matrix $A = A(G)$ is the 0–1 $n \times n$–matrix indexed by the vertices of $G$ and defined by $a_{i,j} = 1$ if and only if $ij \in E$
- The (adjacency) spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $G$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, is the $A$'s spectrum

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

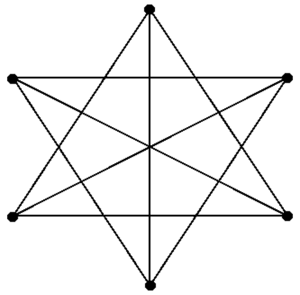
$A$–spectrum : $(3, 1, 0, 0, -2, -2)$



## Laplacian matrix

- The Laplacian of $G$ is defined by $L = L(G) = Deg - A$, where $Deg$ is the diagonal matrix whose diagonal entries are the degrees in $G$, and $A$ the adjacency matrix of $G$
- The Laplacian spectrum $(\mu_1, \mu_2, \ldots, \mu_n)$ of $G$, with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$, is the $L$'s spectrum

$$L = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 & 0 \\ 0 & 3 & 0 & -1 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ -1 & -1 & 0 & 3 & 0 & -1 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$



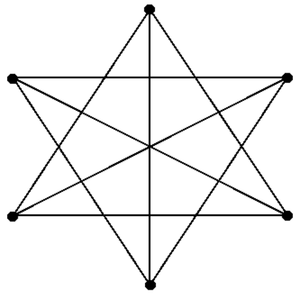
$L$–spectrum : $(5, 5, 3, 3, 2, 0)$

## Signless Laplacian matrix

- The signless Laplacian of $G$ is defined by $Q = Q(G) = Deg + A$
- The Laplacian spectrum $(q_1, q_2, \ldots, q_n)$ of $G$, with $q_1 \geq q_2 \geq \cdots \geq q_n$, is the $Q$'s spectrum

$$Q = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 \\ 1 & 1 & 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 3 \end{bmatrix}$$

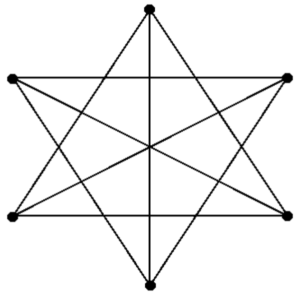
$Q$–spectrum : $(6, 4, 3, 3, 1, 1)$



## Definition

- In a connected graph $G$ the distance $d(i,j) = d_G(i,j)$ is the length of a shortest path between $i$ and $j$
- The distance matrix $\mathcal{D} = \mathcal{D}(G)$ of a connected graph $G$ is the $n \times n$–matrix indexed by the vertices of $G$ and where $\mathcal{D}_{i,j} = d(i,j)$
- The distance spectrum or $\mathcal{D}$–spectrum is denoted by $(\partial_1, \partial_2, \ldots, \partial_n)$ with $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$

$$\mathcal{D} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

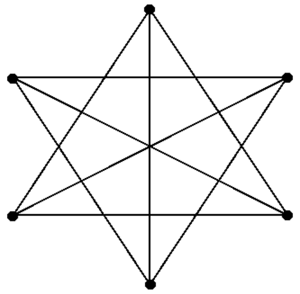


$\mathcal{D}$–spectrum : $(7, 0, 0, -2, -2, -3)$

## Definition

- The transmission of a vertex $i$ is the sum of all the distances from $i$ to all other vertices $t_i = \sum_{j \in V} d(i,j)$.
- The distance Laplacian matrix of $G$ is defined by $\mathcal{D}^L = Tr - \mathcal{D}$, where $Tr$ is the diagonal matrix whose diagonal entries are the transmissions in $G$
- The distance Laplacian spectrum or $\mathcal{D}^L$–spectrum is denoted by $(\partial_1^L, \partial_2^L, \ldots, \partial_n^L)$ with $\partial_1^L \geq \partial_2^L \geq \cdots \geq \partial_n^L = 0$

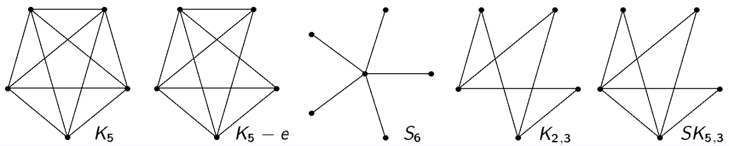
$$\mathcal{D}^L = \begin{bmatrix} 7 & -2 & -1 & -1 & -1 & -2 \\ -2 & 7 & -2 & -1 & -1 & -1 \\ -1 & -2 & 7 & -2 & -1 & -1 \\ -1 & -1 & -2 & 7 & -2 & -1 \\ -1 & -1 & -1 & -2 & 7 & -2 \\ -2 & -1 & -1 & -1 & -2 & 7 \end{bmatrix}$$



$\mathcal{D}^L$–spectrum : $(10, 9, 9, 7, 7, 0)$

## Examples of distance Laplacian spectra

- The complete graph $K_n$ : $\left(n^{(n-1)}, 0\right)$ (also the Laplacian spectrum)
- The complement of an edge $K_n - e$ : $\left(n+2, n^{(n-2)}, 0\right)$
- The star $S_n$ : $\left(2n-1^{(n-2)}, n, 0\right)$
- The complete bipartite graph $K_{a,b}$ : $\left(2n-a^{(b-1)}, 2n-b^{(a-1)}, n, 0\right)$
- The complete split graph $SK_{n,\alpha}$ : $(n+\alpha^{\alpha-1}, n^{n-\alpha}, 0)$



### Properties

- For any connected graph $\partial_n^L = 0$ (with multiplicity $m(0) = 1$)
- $m(\partial_1^L) \leq n-1$, equality holds only for $K_n$
- Among trees $\partial_1^L \geq 2n-1$, equality holds only for $S_n$
- For $L$-spectra :

  $$\mu_1(G) \geq \mu_1(G-e) \geq \mu_2(G) \geq \mu_2(G-e) \geq \cdots \geq \mu_n(G) = \mu_n(G-e) = 0$$

  There is no similar result for $\mathcal{D}^L$–spectra
- The $\mathcal{D}^L$–spectra of $P_6$ and $C_6$ are $(21.3929, 15, 12.8532, 11, 9.7539, 0)$ and $(13, 13, 10, 9, 9, 0)$, respectively
- If $e$ is an edge in $G$ such that $G-e$ is connected, then $\partial_i^L(G-e) \geq \partial_i^L(G)$, for $i = 1, 2, \ldots, n$
- $\partial_i^L(G) \geq \partial_i^L(K_n) = n$, for $i = 1, 2, \ldots n-1$

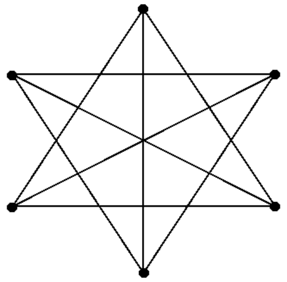
## Transmission regular graphs

A connected graph $G$ is $k$–transmission regular if $t_i = k$, for $i = 1, 2, \cdots, n$
If $G$ is $k$–transmission regular with $\mathcal{D}$–spectrum $(\partial_1, \partial_2, \ldots, \partial_n)$, then $(k - \partial_n, \ldots k - \partial_1)$ is the $\mathcal{D}^L$–spectrum of $G$
Moreover, the eigenspaces are the same

A 7–transmission regular regular

$\mathcal{D}$–spectrum : $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$–spectrum : $(10, 9, 9, 7, 7, 0)$



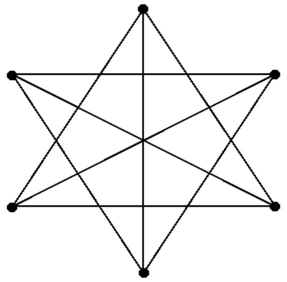
## Graphs of diameter 2

Let $G$ be a graph of diameter $D = 2$, $(\mu_1, \mu_2, \ldots, \mu_n = 0)$ its $L$-spectrum and $(\partial_1, \partial_2, \ldots, \partial_n = 0)$ its $\mathcal{D}^L$–spectrum. Then $\partial_i = 2n - \mu_{n-i}$, for $i = 1, 2, \ldots, n-1$. Moreover, the $L$–eigenspaces and $\mathcal{D}$–eigenspaces coincide

A 7–transmission regular regular

$L$–spectrum : $(5, 5, 3, 3, 2, 0)$

$\mathcal{D}^L$–spectrum : $(10, 9, 9, 7, 7, 0)$



## Similarities with the algebraic connectivity

For the Laplacian $L$ [Fiedler, 1973] :

- $\mu_{n-1} = 0$ if and only if $G$ is disconnected
- The multiplicity of 0 in the $L$–spectrum of $G$ equals the number of connected components of $G$
- $\mu_{n-1}$ is called algebraic connectivity

For the distance Laplacian $\mathcal{D}^L$ :

- $n$ is a $\mathcal{D}^L$–eigenvalue of $G$ if and only if the complement $\overline{G}$ is disconnected
- The multiplicity of $n$ in the $\mathcal{D}^L$–spectrum of $G$ is 1 less than the number of connected components of $\overline{G}$

## Similarities with the algebraic connectivity

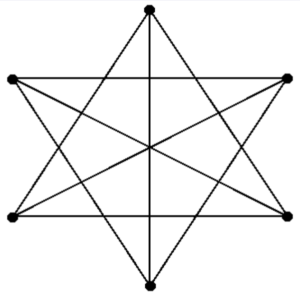
Corollaries :

- $\partial_1(G) \geq n$ with equality if and only if $G \cong K_n$
- If $G$ is bipartite and $n$ is a distance Laplacian eigenvalue of $G$, then $G$ is complete bipartite
- The star $S_n$ is the only tree for which $n$ is a distance Laplacian eigenvalue
- If the maximum degree $\Delta = n - 1$, then $n$ is a $\mathcal{D}^L$–eigenvalue with multiplicity at least $n_\Delta$ (number of vertices of degree $\Delta$)

## Definition

- The transmission of a vertex $i$ is the sum of all the distances from $i$ to all other vertices $t_i = \sum_{j \in V} d(i,j)$.
- The distance Laplacian matrix of $G$ is defined by $\mathcal{D}^Q = Tr + \mathcal{D}$, where $Tr$ is the diagonal matrix whose diagonal entries are the transmissions in $G$
- The distance Laplacian spectrum or $\mathcal{D}^Q$–spectrum is denoted by $(\partial_1^Q, \partial_2^Q, \ldots, \partial_n^Q)$ with $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$

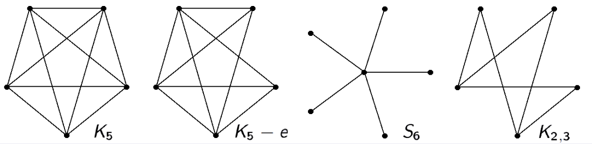
$$\mathcal{D}^Q = \begin{bmatrix} 7 & 2 & 1 & 1 & 1 & 2 \\ 2 & 7 & 2 & 1 & 1 & 1 \\ 1 & 2 & 7 & 2 & 1 & 1 \\ 1 & 1 & 2 & 7 & 2 & 1 \\ 1 & 1 & 1 & 2 & 7 & 2 \\ 2 & 1 & 1 & 1 & 2 & 7 \end{bmatrix}$$



$\mathcal{D}^Q$–spectrum : $(14, 7, 7, 5, 5, 4)$

## Examples of distance signless Laplacian spectra

- For $K_n$ : $\left(2n-2, n-2^{(n-1)}\right)$ (also the signless Laplacian spectrum)
- For $K_n - e$ : $\left(\frac{3n-2\pm\sqrt{(n-2)^2+16}}{2}, n-2^{(n-2)}\right)$
- For $S_n$ : $\left(\frac{5n-8\pm\sqrt{9n^2-32n+32}}{2}, 2n-5^{(n-2)}\right)$
- For $K_{a,b}$ : $\left(\frac{5n-8\pm\sqrt{9(a-b)^2+4ab}}{2}, 2n-a-4^{(b-1)}, 2n-b-4^{(a-1)}\right)$



## Properties

- For $Q$-spectra :

$$q_1(G) \geq q_1(G-e) \geq q_2(G) \geq q_2(G-e) \geq \cdots \geq q_n(G) \geq q_n(G-e)$$

  There is no similar result for $\mathcal{D}^Q$–spectra

- The $\mathcal{D}^Q$–spectra of $P_6$ and $C_6$ are $(25.0838, 12.1755, 11.1743, 8.6727, 7.7418, 5.5118)$ and $(18, 9, 9, 8, 5, 5)$, respectively
- If $e$ is an edge in $G$ such that $G-e$ is connected, then $\partial_i^Q(G-e) \geq \partial_i^Q(G)$, for $i = 1, 2, \ldots, n$
- $\partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n-2$ with equality if and only if $G \cong K_n$
- $\partial_i^Q(G) \geq \partial_i^Q(K_n) = n-2$, for $i = 2, 3, \ldots, n$
- $\partial_2^Q(G) \geq n-2$ with equality if and only if $G \cong K_n$

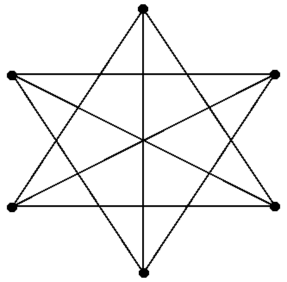
## Transmission regular graphs

- $2Tr_{min} \leq 2\overline{Tr} \leq \partial_1^Q(G) \leq 2Tr_{max}$ with equalities if and only if $G$ is a transmission regular graph
- If $G$ is $k$–transmission regular with $\mathcal{D}$–spectrum $(\partial_1, \partial_2, \ldots, \partial_n)$, then $(k + \partial_1, k + \partial_2, \ldots, k + \partial_n)$ is the $\mathcal{D}^Q$–spectrum of $G$
  Moreover, the eigenspaces are the same

A 7–transmission regular regular

$\mathcal{D}$–spectrum : $(7, 0, 0, -2, -2, -3)$

$\mathcal{D}^L$–spectrum : $(14, 7, 7, 5, 5, 4)$



## Bipartite components

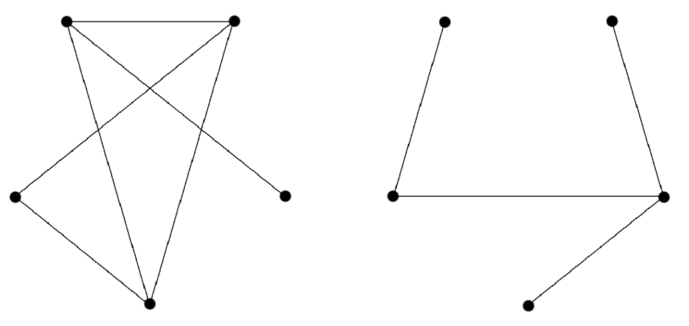
For the signless Laplacian :

- 0 is a $Q$–eigenvalue of $G$ if and only if $G$ contains a bipartite component or an isolated vertex
- The multiplicity of 0 is equal to the number of bipartite components and isolated vertices

For the distance signless Laplacian :

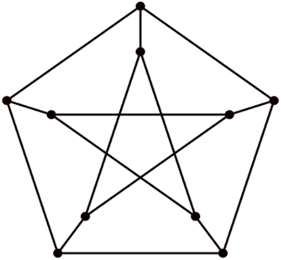
- If $n-2$ is a $\mathcal{D}^Q$–eigenvalue of $G$ with multiplicity $m$, then $\overline{G}$ contains at least $m$ components, each of which is bipartite or an isolated vertex
- There exist graphs with a bipartite complement for which $n-2$ is not a $\mathcal{D}^Q$–eigenvalue

## Bipartite components



$G$ (left) with $n = 5$, $\partial_5^Q \simeq 3.050286 > 3$ and $\overline{G}$ (right) bipartite

## The Petersen graph and its spectra



| | | | |
|---|---|---|---|
| $A$–spectrum | $3^{(1)}$ | $1^{(5)}$ | $-2^{(4)}$ |
| $L$–spectrum | $5^{(4)}$ | $2^{(5)}$ | $0^{(1)}$ |
| $Q$–spectrum | $2^{(1)}$ | $4^{(5)}$ | $1^{(4)}$ |
| $\mathcal{D}$–spectrum | $15^{(1)}$ | $0^{(4)}$ | $-3^{(5)}$ |
| $\mathcal{D}^L$–spectrum | $18^{(5)}$ | $15^{(4)}$ | $0^{(1)}$ |
| $\mathcal{D}^Q$–spectrum | $30^{(1)}$ | $15^{(4)}$ | $12^{(5)}$ |