Forbidden Families of Configurations

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Joint work with Christina Koch
CanaDAM 2013
Memorial University, St. John's, Newfoundland
June 13, 2013

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Consider the following family of subsets of $\{1, 2, 3, 4\}$:

$$\mathcal{A} = \{\emptyset, \{1, 2, 4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}$$

The incidence matrix A of the family $\mathcal{A}$ of subsets of $\{1, 2, 3, 4\}$ is:

$$A = \begin{bmatrix} 0 & 1 & \boxed{1} & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Definition We say that a matrix A is *simple* if it is a (0,1)-matrix with no repeated columns.

Definition We define $\|A\|$ to be the number of columns in A.

$$\|A\| = 6 = |\mathcal{A}|$$

Definition Given a matrix F, we say that A has F as a *configuration* (denoted $F \prec A$) if there is a submatrix of A which is a row and column permutation of F.

$$F = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \prec \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \boxed{1 \quad 0 \quad 1} & & 1 & \boxed{0} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & \boxed{1 \quad 1 \quad 0} & & 0 & \boxed{0} \end{bmatrix}$$

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Definitions

$\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$

$\text{Avoid}(m, \mathcal{F}) = \{\, A \,:\, A$ m-rowed simple, $F \nprec A$ for all $F \in \mathcal{F}\}$

$forb(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}$

Main Bounds

Definition Let K_k be the $k \times 2^k$ simple matrix of all possible columns on k rows.

Theorem (Sauer 72, Perles and Shelah 72, Vapnik and Chervonenkis 71)

$$forb(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} \text{ which is } \Theta(m^{k-1}).$$

Theorem (Füredi 83). *Let F be a $k \times \ell$ matrix. Then $forb(m, F) = O(m^k)$.*

Problem Given $\mathcal{F}$, can we predict the behaviour of $forb(m, \mathcal{F})$?

Balanced and Totally Balanced Matrices

Let C_k denote the $k \times k$ vertex-edge incidence matrix of the cycle of length k.

$$\text{e.g.} \quad C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

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Matrices in Avoid$(m, \{C_3, C_5, C_7, \ldots\})$ are called
Balanced Matrices.
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Matrices in Avoid$(m, \{C_3, C_4, C_5, C_6, \ldots\})$ are called
Totally Balanced Matrices.
Theorem $\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \mathit{forb}(m, C_3)$

Remark If $\mathcal{F}' \subset \mathcal{F}$ then $\mathit{forb}(m, \mathcal{F}) \leq \mathit{forb}(m, \mathcal{F}')$.

The inequality $\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) \leq \mathit{forb}(m, C_3)$ follows from the remark.

The equality follows from a result that any $m \times \mathit{forb}(m, C_3)$ simple matrix in Avoid(m, C_3) is in fact totally balanced (A, 80).
Thus we conclude
$\mathit{forb}(m, \{C_3, C_4, C_5, C_6, \ldots\}) = \mathit{forb}(m, C_3)$.

A Product Construction

The building blocks of our product constructions are I, I^c and T:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_4^c = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition Given an $m_1 \times n_1$ matrix A and a $m_2 \times n_2$ matrix B we define the product $A \times B$ as the $(m_1 + m_2) \times (n_1 n_2)$ matrix consisting of all $n_1 n_2$ possible columns formed from placing a column of A on top of a column of B. If A, B are simple, then $A \times B$ is simple. (A, Griggs, Sali 97)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Given p simple matrices $A_1, A_2, \ldots, A_p$, each of size $m/p \times m/p$, the p-fold product $A_1 \times A_2 \times \cdots \times A_p$ is a simple matrix of size $m \times (m^p/p^p)$ i.e. $\Theta(m^p)$ columns.

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The Conjecture

Definition Let $x(\mathcal{F})$ denote the smallest p
such that for every p-fold product $A_1 \times A_2 \times \cdots \times A_p$,
where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$,
there is some $F \in \mathcal{F}$ with $F \prec A_1 \times A_2 \times \cdots \times A_p$.
Thus there is some $(p-1)$-fold product
$A_1 \times A_2 \times \cdots \times A_{p-1} \in \text{Avoid}(m, \mathcal{F})$
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Conjecture (A, Sali 05) *Let* $|\mathcal{F}| = 1$*. Then* $forb(m, \mathcal{F})$ *is* $\Theta(m^{x(\mathcal{F})-1})$*.*

In other words, we predict our product constructions with the three building blocks $\{I, I^c, T\}$ determine the asymptotically best constructions when $|\mathcal{F}| = 1$.

The Conjecture

Definition Let $x(\mathcal{F})$ denote the smallest p such that for every p-fold product $A_1 \times A_2 \times \cdots \times A_p$, where each $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$, there is some $F \in \mathcal{F}$ with $F \prec A_1 \times A_2 \times \cdots \times A_p$.
Thus there is some $(p-1)$-fold product $A_1 \times A_2 \times \cdots \times A_{p-1} \in \text{Avoid}(m, \mathcal{F})$ showing that $forb(m, \mathcal{F})$ is $\Omega(m^{p-1})$.

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The conjecture has been verified for $k \times \ell$ F where $k = 2$ (A, Griggs, Sali 97) and $k = 3$ (A, Sali 05) and $\ell = 2$ (A, Keevash 06).

Forbidden Families can fail Conjecture

Definition ex(m, H) is the maximum number of edges in a (simple) graph G on m vertices that has no subgraph H.

$A \in \text{Avoid}(m, \mathbf{1}_3)$ will be a matrix with up to $m+1$ columns of sum 0 or sum 1 plus columns of sum 2 which can be viewed as the vertex-edge incidence matrix of a graph.
Let Inc(H) denote the $|V(H)| \times |E(H)|$ vertex-edge incidence matrix associated with H.
Theorem $forb(m, \{\mathbf{1}_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H)$.

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Theorem $\mathit{forb}(m, \{\mathbf{1}_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H)$.

In this talk $I(C_4) = C_4$, $I(C_6) = C_6$.

Theorem $\mathit{forb}(m, \{\mathbf{1}_3, C_4\}) = m + 1 + \text{ex}(m, C_4)$ which is $\Theta(m^{3/2})$. note that $x(\{\mathbf{1}_3, C_4\}) = 2$

Theorem $\mathit{forb}(m, \{\mathbf{1}_3, C_6\}) = m + 1 + \text{ex}(m, C_6)$ which is $\Theta(m^{4/3})$. note that $x(\{\mathbf{1}_3, C_6\}) = 2$

Forbidden Families can pass Conjecture

Theorem $forb(m, \{\mathbf{1}_3, \text{Inc}(H)\}) = m + 1 + \text{ex}(m, H)$.

Theorem Let T be a graph with no cycles. Then ex(m, T) is $O(m)$.

Corollary Let F be a (0,1)-matrix with column sums at most 2. Assume $C_k \not\prec F$ for $k = 2, 3, \ldots$ (we don't allow repeated columns of sum 2 but allow other repeated columns). Then $forb(m, \{\mathbf{1}_3, F\})$ is $O(m)$.

Proof: We can find a graph T with no cycles such that $F \prec \text{Inc}(T)$. Then $forb(m, \{\mathbf{1}_3, F\}) \leq m + 1 + \text{ex}(m, T)$.

Forbidden Families can pass Conjecture

Theorem (Balogh and Bollobás 05) Let k be given. Then there is a constant c_k so that $forb(m, \{I_k, I_k^c, T_k\}) = c_k$.

We note that $x(\{I_k, I_k^c, T_k\}) = 1$ and so there is no obvious product construction.

Note that $c_k \geq \binom{2k-2}{k-1}$ by taking all columns of column sum at most $k-1$ that arise from the $k-1$-fold product $T_{k-1} \times T_{k-1} \times \cdots \times T_{k-1}$.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$.

Lemma Let $\mathcal{F}$ and $\mathcal{G}$ have the property that for every $G_i \in \mathcal{G}$, there is some $F_j \in \mathcal{F}$ with $F_j \prec G_i$. Then $\mathit{forb}(m, \mathcal{F}) \leq \mathit{forb}(m, \mathcal{G})$.

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Theorem Let $\mathcal{F}$ be given. Then either $\mathit{forb}(m, \mathcal{F})$ is $O(1)$ or $\mathit{forb}(m, \mathcal{F})$ is $\Omega(m)$.

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Theorem Let $\mathcal{F}$ be given. Then either $forb(m, \mathcal{F})$ is $O(1)$ or $forb(m, \mathcal{F})$ is $\Omega(m)$.

Proof: We start using $\mathcal{G} = \{I_p, I_p^c, T_p\}$ with p suitably large.
Either
we have the property that there is some $F_r \prec I_p$, and some $F_s \prec I_p^c$ and some $F_t \prec T_p$ in which case $forb(m, \mathcal{F}) \leq forb(m, \{I_p, I_p^c, T_p\})$ which is $O(1)$
or
without loss of generality we have $F_j \not\prec I_p$ for all j and hence $I_m \in \text{Avoid}(m, \mathcal{F})$ and so $forb(m, \mathcal{F})$ is $\Omega(m)$.

A pair of Configurations with quadratic bounds

e.g. $F_2(1,2,2,1) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \not\prec I \times I^c$.

$$\underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \times \underset{I_3^c}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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$I_{m/2} \times I^c_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding $F_2(1,2,2,1)$, so $forb(m, F_2(1,2,2,1))$ is $\Omega(m^2)$.

(A, Ferguson, Sali 01 $forb(m, F_2(1,2,2,1)) = \lfloor \frac{m^2}{4} \rfloor + \binom{m}{1} + \binom{m}{0}$)

A pair of Configurations with quadratic bounds

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\prec T \times T$. Also $I_3 \not\prec I^c \times T$, $I_3 \not\prec I^c \times I^c$

$$\underset{T_3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \times \underset{T_3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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$T_{m/2} \times T_{m/2}$ is an $m \times m^2/4$ simple matrix avoiding I_3,
so $forb(m, I_3)$ is $\Omega(m^2)$.
$(forb(m, I_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0})$

Forbidden Families can pass Conjecture

By considering the construction $I \times I^c$ that avoids $F_2(1,2,2,1)$ and the constructions $I^c \times I^c$ or $I^c \times T$ or $T \times T$ that avoids I_3, we note $x(\{I_3, F_2(1,2,2,1)\}) = 2$ so that we have only linear obvious constructions (I_m^c or T_m) that avoid both $F_2(1,2,2,1)$ and I_3. We are led to the following:

Theorem $\mathit{forb}(m, \{I_3, F_2(1,2,2,1)\})$ is $\Theta(m)$.

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Theorem $forb(m, \{I_3, F_2(1,2,2,1)\})$ is $\Theta(m)$.

We can extend the argument quite far:

Theorem $forb(m, \{t \cdot I_k, F_2(1,t,t,1)\})$ is $\Theta(m)$.

Another example:

$$forb\left(m, \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \overbrace{1\,1\cdots 1}^{t} & \overbrace{0\,0\cdots 0}^{t} & 1 \\ 0 & 0\,0\cdots 0 & 1\,1\cdots 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0\,0\cdots 0 & 1\,1\cdots 1 & 1 \end{bmatrix} \right\}\right) \text{ is } O(m).$$

Another example:

$$forb\left(m,\left\{\begin{bmatrix}1&0\\1&0\\0&1\\0&1\end{bmatrix},\begin{bmatrix}0&\overbrace{1\,1\cdots1}^{t}&\overbrace{0\,0\cdots0}^{t}&1\\0&0\,0\cdots0&1\,1\cdots1&1\\\vdots&\vdots&\vdots&\vdots\\0&0\,0\cdots0&1\,1\cdots1&1\end{bmatrix}\right\}\right)\text{ is }O(m).$$

We studied the 9 'minimal' configurations that have quadratic bounds and were able to verify the predictions of the conjecture for all subsets of these 9.

An unusual Bound

Theorem (A,Koch,Raggi,Sali 12) $forb(m, \{T_2 \times T_2, I_2 \times I_2\})$ *is* $\Theta(m^{3/2})$.

$$T_2 \times T_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, I_2 \times I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (= C_4)$$

We showed initially that $forb(m, \{T_2 \times T_2, T_2 \times I_2, I_2 \times I_2\})$ is $\Theta(m^{3/2})$ but Christina Koch realized that we ought to be able to drop $T_2 \times I_2$ and we were able to redo the proof (which simplified slightly!).


Miguel Raggi, Attila Sali

Induction

Let A be an $m \times forb(m, \mathcal{F})$ simple matrix with no configuration in $\mathcal{F} = \{T_2 \times T_2, I_2 \times I_2\}$. We can select a row r and reorder rows and columns to obtain

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \left[\begin{array}{cccccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r & & C_r & C_r & & D_r \end{array} \right].$$

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To show $\|A\|$ is $O(m^{3/2})$ it would suffice to show $\|C_r\|$ is $O(m^{1/2})$ for some choice of r. Our proof shows that assuming $\|C_r\| > 20m^{1/2}$ for all choices r results in a contradiction. In particular, associated with C_r is a set of rows $S(r)$ with $S(r) \geq 5m^{1/2}$. We let $S(r) = \{r_1, r_2, r_3, \ldots\}$. After some work we show that $|S(r_i) \cap S(r_j)| \leq 5$. Then we have
$|S(r_1) \cup S(r_2) \cup S(r_3) \cup \cdots|$
$= |S(r_1)| + |S(r_2) \backslash S(r_1)| + |S(r_3) \backslash (S(r_1) \cup S(r_2))| + \cdots$
$= 5m^{1/2} + (5m^{1/2} - 5) + (5m^{1/2} - 10) + \cdots > m$! ! !

Thanks to all the organizers of CanaDAM 2013!
Great to visit Newfoundland.
I very much enjoyed the Fish and Brew(i)s.