# Skolem labelled graphs, old and new results

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# Outline

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In this talk we survey the known results about Skolem labelling of graphs.

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- In Progress, David Pike, Asiyeh Sanaei and Nabil Shalaby, Pseudo-Skolem sequences and rail-siding graphs Skolem labelling.

# Definitions and examples

### Definition

A Skolem-type sequence is a sequence $(s_1, s_2, \ldots, s_m)$ of positive integers $i \in D$ where $D$ is a set of positive integers called differences and for each $i \in D$ there is exactly one $j \in \{1, 2, \ldots, m - i\}$ such that $s_i = s_{j+i} = i$. A Skolem sequence of order $n$ is a partition of the set $\{1, 2, \ldots, 2n\}$ into a collection of disjoint ordered pairs $\{(a_i, b_i) : \ i = 1, 2, \ldots, n\}$ such that $a_i < b_i$ and $b_i - a_i = i$. Equivalently, a Skolem sequence is a Skolem-type sequence with $m = 2n$ and $D = \{1, 2, \ldots, n\}$.

### Example

$D = \{1, 2, 3, 4\} \Rightarrow \frac{4}{1}\frac{2}{2}\frac{3}{3}\frac{2}{4}\frac{4}{5}\frac{3}{6}\frac{1}{7}\frac{1}{8}$ (Skolem sequence of order 4)

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$$\text{or } \{(7, 8), (2, 4), (3, 6), (1, 5)\} \text{ or } (4, 2, 3, 2, 4, 3, 1, 1)$$

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or $\{(7, 8), (2, 4), (3, 6), (1, 5)\}$ or $(4, 2, 3, 2, 4, 3, 1, 1)$

$(3, 1, 1, 3, 2, *, 2)$ Is a hooked Skolem sequence of order 3.

# Skolem labelled graphs: E. Mendelsohn, N. Shalaby, 1991

### Definition

A Skolem labelled graph is a triple $(G, L, d)$, where $G = (V, E)$ is a graph and $L : V \rightarrow \{d, d+1, \ldots, d+m\}$ satisfying:

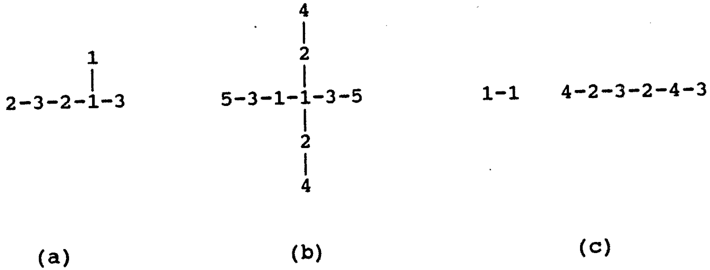
1. There are exactly two vertices in $V$, such that $L(v) = d + i$, $0 \leq i \leq m$.
2. The distance in $G$ between any two vertices with the same label is the value of the label.
3. If $G' = (V, E')$ and $E' \overset{\subset}{\neq} E$ then $(G', L, d)$ violates (2).

# Definitions and examples

## Remark

Given a Skolem sequence of order 4, $4, 1, 1, 3, 4, 2, 3, 2$, it is natural to think of $4 - 1 - 1 - 3 - 4 - 2 - 3 - 2$ as a labelling of a 7-path.

## Example

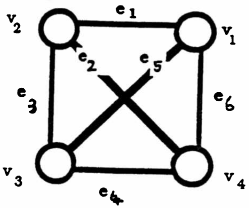


# Embedding Theorem: E. Mendelsohn, N. Shalaby, 1991

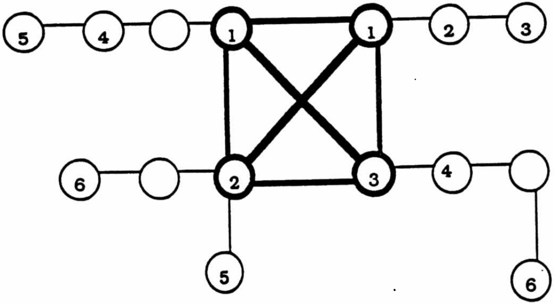
**Theorem**

*Every graph with* $v$ *vertices and* $e$ *edges can be embedded in a Skolem labelled graph with* $O(v^3)$

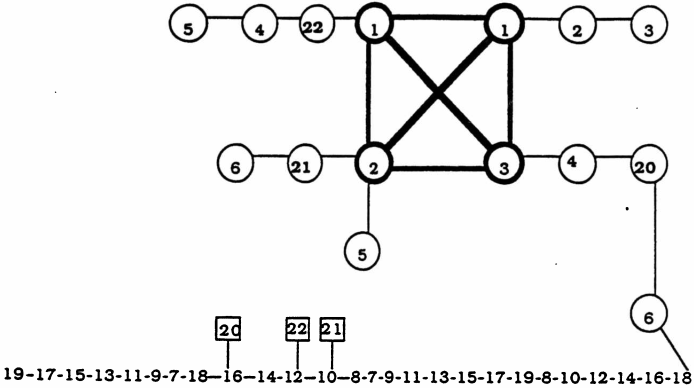
An Embedding of $K_4$



$K_4$



EMBEDDING STAGE 1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19-17-15-13-11-9-7-18—16—14-12—10—8-7-9-11-13-15-17-19-8-10-12-14-16-18

$K_4$ EMBEDDED

# Outline of Proof

**After Stage 1 of the embedding** :

The Number of added unlabelled vertices is :

$$\leq \frac{v(v+1)(2v-5)}{6}$$

**After Stage 2 of the embedding**

The total number of vertices of the embedding is :

$$= \frac{v(v+1)(2v-5)}{3} + 3v(v+1) + 2$$

# On Skolem labelling of Windmills: E. Mendelsohn, N. Shalaby, 1999

### Remark

Here it is proved that the necessary conditions are sufficient for a Skolem or minimum hooked Skolem labelling of all windmills. A $k$-windmill is a tree with $k$ leaves each lying on an edge-disjoint path of length $m$, to the centre. These paths are called the vanes.

### Theorem

*A necessary condition for the existence of a Skolem labelling of any tree $T$ with $2n$ vertices are:*

1. *If $n \equiv 0, 3 \pmod 4$ the parity of $T$ must be even.*
2. *If $n \equiv 1, 2 \pmod 4$ the parity of $T$ must be odd.*

# Skolem arrays and Skolem labellings of ladder graphs: C. Baker, P. Kergin, A. Bonato, 2002

### Remark

Here Skolem arrays are introduced, which are two-dimensional analogues of Skolem sequences. Skolem arrays are ladders which admit a Skolem labelling. They proved that they exist exactly for those integers $n \equiv 0$ or $1 \pmod 4$. In addition, they provided an exponential lower bound for the number of distinct Skolem arrays of a given order. Computational results were presented which give an exact count of the number of Skolem arrays up to order 16.

### Definition

A Skolem array is a $2 \times n$ array $A$ in which each $i \in \{1, 2, \ldots, n\}$ occurs in two positions of A which are distance $i$ apart.

### Example

Skolem array of order 1:

|   |
|---|
| 1 |
| 1 |

### Example

Skolem array of order 4:

|   |   |   |   |
|---|---|---|---|
| 3 | 1 | 1 | 4 |
| 4 | 2 | 3 | 2 |

# Skolem labelled and $P_s \Box P_t$ Cartesian Products: A. Graham, D. Pike, N. Shalaby, 2007

### Remark

Here the necessary and sufficient conditions were proved for maximum (hooked) Skolem labelling $P_s \Box P_t$ Cartesian Products. They also provided an algorithm used to generate Skolem labellings of trees as well as data generated using this algorithm.

### Example

|   |   |   |
|---|---|---|
| 4 | 1 | 1 |
| 2 | * | 3 |
| 3 | 2 | 4 |

is a hooked Skolem labelling for $P_3 \Box P_3$.

# Skolem labelling of Generalized Three-Vane Windmills: C. Baker, J. Manzer, 2008

**Remark**

Here the authors removed the restriction that the vanes must have equal length and considered generalized $k$-windmills.

**Definition**

A $k$-windmill is a tree consisting of $k$ paths of equal positive length, called *vanes*, which meet at a central vertex called the *pivot*.

**Definition**

A generalized $k$-windmill is a windmill in which the $k$ vanes may be of different positive lengths.

### Example

$$\begin{array}{cccccccccccccc}
9 & 7 & 5 & 3 & 1 & 1 & 3 & 5 & 7 & 9 & 2 & 4 & 6 & 8 \\
 & & & & & & & & & 2 & & & & \\
 & & & & & & & & & 4 & & & & \\
 & & & & & & & & & 6 & & & & \\
 & & & & & & & & & 8 & & & &
\end{array}$$

is a $W(9:9,4,4)$.

### Example

$$\begin{array}{ccccccc}
4 & 1 & 1 & 3 & 4 & 2 & 3 \\
 & & & & 2 & &
\end{array}$$

is a $W(4:4,2,1)$.

# Pseudo-Skolem sequences and Rail-siding Graphs Skolem labelling: D. Pike, A. Sanaei, N. Shalaby

### Remark

They introduce pseudo-Skolem sequences which are similar to Skolem sequences not only in their structure but also in their helpfulness. Then they demonstrate the use of these sequences to Skolem label graphs in particular classes of rail-siding graphs and caterpillars.

### Remark

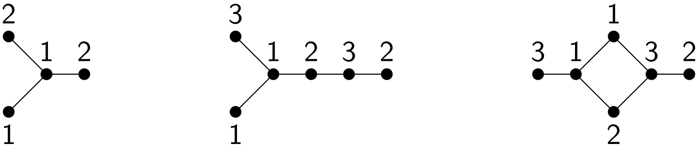
In a Skolem sequence the pairs $(a_i, b_i)$ are disjoint for $i \in \{1, 2, \dots, n\}$. If we allow some of the pairs to share a point, then we have what we call a *pseudo-Skolem sequence*.

### Definition

Suppose that $\{k, n\} \subset \mathbb{N}$ such that $1 \leq k \leq 2n - 1$. A *k-pseudo-Skolem sequence* of order $n$ is a distribution of the elements of the set $\{1, 2, \dots, 2n-1\}$ into a collection of ordered pairs $\{(a_i, b_i) : \; i = 1, 2, \dots, n\}$ such that $a_i < b_i$ and $b_i - a_i = i$ and the pairs that do not contain $k$ are mutually disjoint (there are exactly two pairs containing $k$). We may show a $k$-pseudo-Skolem sequence with $(s_1, s_2, \dots, s_{k-1}, \overset{s'_k}{s_k}, s_{k+1}, \dots, s_{2n-1})$ of positive integers $i \in \{1, 2, \dots, n\}$ such that for each $i$ there is exactly one $j \in \{1, 2, \dots, 2n-1-i\}$ such that $s_i = s_{j+i} = i$, $s'_i = s_{j+i} = i$, or $s_i = s'_{j+i} = i$. The integer $k$ is called the pocket of the sequence.

### Example

If $n = 2$, then $\{(1,2),(1,3)\}$ or equivalently $(\overset{2}{1},1,2)$ is a 1-pseudo-Skolem sequence of order 2 with 1 being the pocket of the sequence.
If $n = 3$, then $\{(1,2),(3,5),(1,4)\}$ and $\{(2,3),(3,5),(1,4)\}$ (or equivalently $(\overset{3}{1},1,2,3,2)$ and $(3,1,\overset{2}{1},3,2)$) are 1-pseudo-Skolem and 3-pseudo-Skolem sequences of order 3 (resp.), and with 1 and 3 being the pockets of the sequences (resp.). Note that these three pseudo-Skolem sequences are equivalent to Skolem labellings of the rail-siding graphs.



### Remark

Similarly, we can define pseudo-Skolem sequences with $p$ pockets for every $p \geq 2$.

### Definition

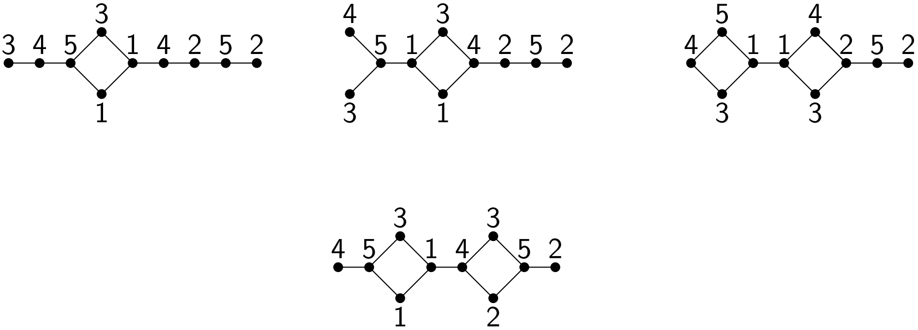
Suppose that $\{k_1, k_2, \ldots, k_p, n\} \subset \mathbb{N}$ such that $1 \leq k_i \leq 2n - p$. A $\{k_1, k_2, \ldots, k_p\}$-*pseudo-Skolem sequence* of order $n$ is a distribution of the set $\{1, 2, \ldots, 2n - p\}$ into a collection of ordered pairs $\{(a_i, b_i) : \ i = 1, 2, \ldots, n\}$ such that $a_i < b_i$ and $b_i - a_i = i$ and the pairs that do not contain $k_i$, $1 \leq i \leq p$, are mutually disjoint (there are exactly $p$ pairs with $k_i$, $1 \leq i \leq p$, as an element). We may show a $\{k_1, k_2, \ldots, k_p\}$-pseudo-Skolem sequence with $(s_1, s_2, \ldots, \overset{s'_{k_1}}{s_{k_1}}, s_{k_1+1}, \ldots, \overset{s'_{k_2}}{s_{k_2}}, s_{k_2+1}, \ldots, \overset{s'_{k_p}}{s_{k_p}}, s_{k_p+1}, \ldots, s_{2n-p})$ of positive integers $i \in \{1, 2, \ldots, n\}$ such that for each $i$ there is exactly one $j \in \{1, 2, \ldots, 2n - p - i\}$ such that $s_i = s_{j+i} = i$, $s'_i = s_{j+i} = i$, $s'_i = s'_{j+i} = i$, or $s_i = s'_{j+i} = i$. The integers $k_i$ for $1 \leq i \leq p$ are called the pockets of the sequence.

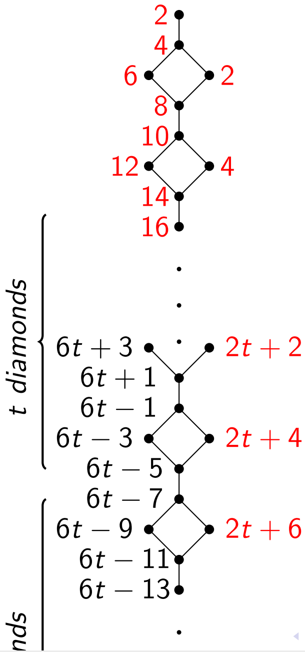
### Example

If $n = 4$, then $\{(2,3),(4,6),(3,6),(1,5)\}$ or equivalently $(4,1,\overset{3}{1},2,4,\overset{3}{2})$ is a $\{3,6\}$-pseudo-Skolem sequence of order 4 with 3 and 6 being the pockets of the sequence.

In this paper they show that using the known Skolem-type sequences we can obtain pseudo-Skolem sequences and thereby Skolem label rail-siding graphs.

For example, having 3-near Skolem sequence of order 5 (4,5,1,1,4,2,5,2) we can have pseudo-Skolem sequences $(3,4,5,\overset{3}{1},1,4,2,5,2)$, $(\overset{3}{4},5,1,\overset{3}{1},4,2,5,2)$, $(4,\overset{3}{5},1,1,\overset{3}{4},2,5,2)$, $(4,5,\overset{3}{1},1,4,\overset{3}{2},5,2)$ and hence Skolem labelling for the graphs below.

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6t + 1
6t − 1
6t − 3
2t + 4
6t − 5
6t − 7
6t − 9
2t + 6
6t − 11
6t − 13
nds

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- Find the necessary degeneracy condition for the Skolem labelling of trees.

THANK YOU!...