Generalised n-gons with symmetry conditions

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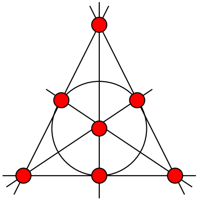

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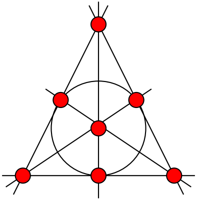

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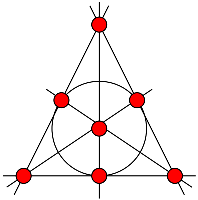

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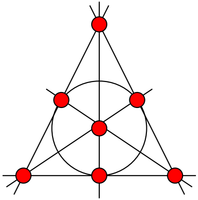

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Feit-Higman (1964): finite and thick implies $n \in \{2, 3, 4, 6, 8\}$.

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Many other examples of projective planes and generalised quadrangles known.

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- K. Thas and Zagier (2008): A non-Desarguesian, flag-transitive plane has at least 4×10^{22} points.

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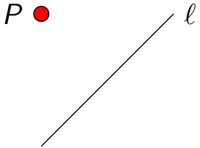
Note: There are two known non-classical generalised quadrangles that are flag-transitive, with $(s,t) = (3,5)$ and $(s,t) = (15,17)$. The flag-transitive group actions on these are point-primitive but not line-primitive, so the line-primitivity condition is important.

Generalised quadrangles

A generalised quadrangle is a point-line incidence geometry $\mathcal{Q}$ such that:

1. any two points lie on at most one line, and
2. given a line ℓ and a point P not incident with ℓ, P is collinear with a unique point of ℓ.



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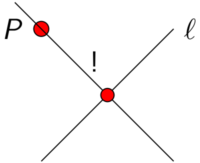

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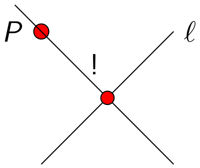

A GQ of order (s, t) has $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines.

O'Nan-Scott Theorem

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The possible O'Nan-Scott types for two faithful primitive actions of a group are:

Primitive type on Ω_1	Primitive type on Ω_2	Comments
Almost Simple	Almost Simple	
HA (affine)	HA	$\vert\Omega_1\vert = \vert\Omega_2\vert = p^d$
HS	HS	$\vert\Omega_1\vert = \vert\Omega_2\vert = \vert T\vert$
HC	HC	$\vert\Omega_1\vert = \vert\Omega_2\vert = \vert T\vert^k$
TW	TW, SD, CD, PA	
SD	TW, SD, PA	
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- Numerical conditions about generalised quadrangles can be used to show that if $\gcd(s, t) > 1$ and θ is a fixed-point-free involution of the GQ, then θ must fix a line of the GQ.
- The Feit-Thompson theorem can be used to produce fixed-point-free involutions in our primitive group (acting on points) that (by properties of the O'Nan-Scott types) cannot fix a line.

These results help eliminate many cases.

Primitive type on points	Primitive type on lines
AS	AS
~~HA~~	~~HA~~
~~HS~~	~~HS~~
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Note: The dual (interchange points and lines) of a generalised *n*-gon is a generalised *n*-gon.

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Note: Using this to rule out TW requires the Schreier conjecture, so the Classification of Finite Simple Groups is being used in this proof.

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It is known that having two properly nested substructures creates severe restrictions on the numbers of points and lines of each. This is sufficient to rule out this possibility.

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Only possibility is Ru with $s = t = 57$ which can be eliminated.

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If flag-transitive, $|G| \leq |G_P|^6$. Bounds on orders of primitive groups then imply $n \leq 47$. These cases can be checked.

Thank you!