Cycle decompositions of complete multigraphs

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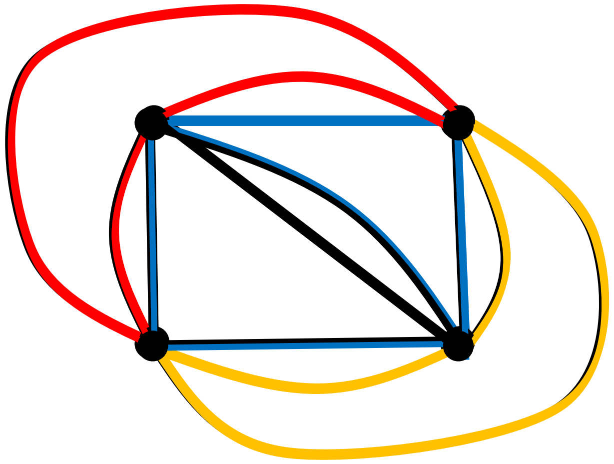
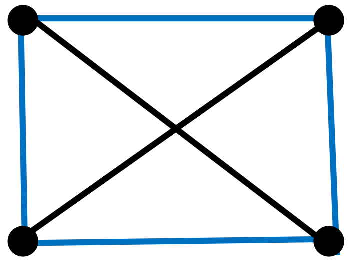
Cycle decompositions of the complete multigraph λK_n

Example: $2K_4$





A decomposition of λK_n into cycles is a set of cycles that are subgraphs of λK_n whose edge sets partition the edge set of λK_n.

Obvious requirements:

- the sum of the cycle lengths be equal to the number of edges in λK_n;
- the degree of each vertex of λK_n to be even.

In the case, where the degree is not even, we decompose λK_n into cycles and a perfect matching.

Cycle decompositions of the complete multigraph λK_n : necessary conditions

For $n \geq 3, \lambda \geq 1$, to partition the edge set of λK_n into *t* cycles of lengths $m_1, m_2, \ldots, m_t$, or into *t* cycles of lengths $m_1, m_2, \ldots, m_t$ and a perfect matching, we require that:

- $2 \leq m_1, m_2, \ldots, m_t \leq n$;
- $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2}$ when $\lambda(n-1)$ is even;
- $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2} - \frac{n}{2}$ when $\lambda(n-1)$ is odd;
- $\sum_{m_i > 2} m_i \geq n\left\lfloor\frac{n-1}{2}\right\rfloor$ when λ is odd.

A list of integers $m_1, m_2, \ldots, m_t$ that satisfy the above conditions for particular values of λ and n is said to be (λ, n)–admissible.

For shorthand, if $M = m_1, m_2, \ldots, m_t$, the notation (*M*)*-decomposition of λK_n will be used to denote both a decomposition of λK_n into *t* cycles of lengths $m_1, m_2, \ldots, m_t$ and a decomposition of λK_n into *t* cycles of lengths $m_1, m_2, \ldots, m_t$ and a perfect matching.

Cycle decompositions of the complete multigraph λK_n : constant length cycles

Theorem: Let λ, n and *m* be integers with $n, m \geq 3$ and $\lambda \geq 1$.
There exists a decomposition of λK_n into cycles of length *m* if and only if

$$m \leq n; \quad \lambda\,(n-1) \;\; is\ even; \;\; and\ m\ divides\ \lambda\,\binom{n}{2}.$$

There exists a decomposition of λK_n into cycles of length *m* and a perfect matching if and only if

$$m \leq n; \quad \lambda\,(n-1) \;\; is\ odd; \;\; and\ m\ divides\ \lambda\,\binom{n}{2} - n/2.$$

A very brief history of the problem of decomposing λK_n into *m*-cycles (or into *m*-cycles and a perfect matching):

- The case $\lambda = 1$: Many specific cases solved over many years, but finally solved by Alspach, Gavlas and Šajna (2001, 2002).
- The case $\lambda = 2$: Solved by Alspach, Gavlas, Šajna and Verall by considering decompositions into directed cycles (2003).
- The cases $\lambda \geq 3$: $m = 3$ (Hanani, 1961),
  $4 \leq m \leq 6$ (Huang and Rosa, 1973, 1975),
  $8 \leq m \leq 16$, *m* even (Bermond, Huang and Sotteau, 1978),
  $3 \leq m \leq 7, m$ odd (Bermond and Sotteau, 1977),
  m an odd prime (Smith, 2010)
  λ a multiple of *m* (Smith, 2010),
  n odd and λn a multiple of *m* (Smith, 2010).

Cycle decompositions of the complete multigraph λK_n : Two theorems for mixed length cycles

Long Cycle Theorem: Let $n \geq 3$ and λ be positive integers and let $M = m_1, m_2, \ldots, m_t$ be a (λ, n)-admissible list of integers. If $m_i \geq \lfloor\frac{n+3}{2}\rfloor$ for $i = 1, 2, \ldots, t$, then there exists an (M)*-decomposition of λK_n.

For example, there exists a (6,6,6,6,6,7,7,7,7,8,9,9,9,9,9,9,9,10)*-decomposition of $3K_{10}$.

Short Cycle Theorem: Let $n \geq 3$ and λ be positive integers and let $M = m_1, m_2, \ldots, m_t$ be a non-decreasing (λ, n)-admissible list of integers such that either $m_t = m_{t-1} \leq \lfloor\frac{n+1}{2}\rfloor$ or $m_t = m_{t-1} + 1 \leq \lfloor\frac{n+2}{2}\rfloor$. Then there exists an ($M$)*-decomposition of λK_n.

For example, there exists a (3,3,3,3,3,3,4,4,4,5,5)*-decomposition of K_{10} and a (3,3,3,3,3,3,3,4,4,5,6)*-decomposition of K_{10}.

Cycle decompositions of the complete multigraph λK_n : Start with decompositions into closed trails

Theorem (Balister) Let λ and *n* be positive integers with $n \geq 3$ and $\lambda(n-1)$ even. There exists a decomposition of λK_n into *t* closed trails of lengths $m_1, m_2, \ldots, m_t$ if and only if

- $2 \leq m_1, m_2, \ldots, m_t$;
- $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2}$; and
- $\sum_{m_i > 2} m_i \geq \binom{n}{2}$ when λ is odd.

Theorem Let λ and *n* be positive integers with $n \geq 4$ and $\lambda(n-1)$ odd. There exists a decomposition of λK_n into *t* closed trails of lengths $m_1, m_2, \ldots, m_t$ and a perfect matching if and only if

- $2 \leq m_1, m_2, \ldots, m_t$;
- $m_1 + m_2 + \cdots + m_t = \lambda\binom{n}{2} - \frac{n}{2}$; and
- $\sum_{m_i > 2} m_i \geq \binom{n}{2} - \frac{n}{2}$.

Plan: For $M = m_1, m_2, \ldots, m_t$, to get an (*M*)*-decomposition of λK_n, we start with a decomposition of λK_n into closed trails of lengths $m_1, m_2, \ldots, m_t$ (and maybe a perfect matching). Split the closed trails into cycles and then manipulate (modify and glue) these cycles in such a way as to get cycles of lengths $m_1, m_2, \ldots, m_t$ (and maybe a perfect matching).

Cycle decompositions of the complete multigraph λK_n : Outline of proof for short cycle theorem

Let $m_1, m_2, \ldots, m_t$ be a non-decreasing (λ, n)-admissible list, in which $m_t = m_{t-1} \leq \lfloor \frac{n+1}{2} \rfloor$ or $m_t = m_{t-1} + 1 \leq \lfloor \frac{n+2}{2} \rfloor$.
To find an $(m_1, m_2, \ldots, m_t)^*$-decomposition of λK_n :

Start with a decomposition of λK_n into closed trails of lengths $m_1, m_2, \ldots, m_t$.
If these all happen to be cycles, you are done.

If the biggest closed trails (lengths m_{t-1} and m_t) are not cycles, delete those two closed trails from the decomposition (they become the leave of a packing) and use edge switches to make them into cycles of lengths m_{t-1} and m_t.
(This can be done since m_{t-1} and m_t differ by at most one.)

Then for each remaining closed trail (say of length m_i) that is not a cycle:

Delete the closed trail from the decomposition (it becomes the leave) and use edge switches to spread it out to a collection of almost vertex-disjoint cycles of lengths $a_1, a_2, \ldots, a_s$, where $a_1 + a_2 + \cdots + a_s = m_i$.

Use edge switches to join the almost vertex-disjoint cycles into a chain of cycles.

Add the cycle of length m_t to the cycle chain as the leave of a packing and use edge switches to obtain a cycle of length m_t and a cycle of length m_i.

Thank you for your attention!