Locally injective homomorphisms

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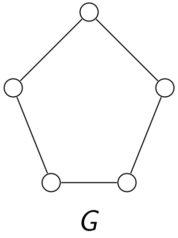
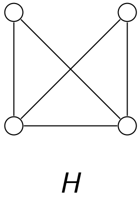
Homomorphisms

For graphs G and H, think of $V(H)$ as a set of colours. Colour $V(G)$ so that adjacent vertices get adjacent colours.





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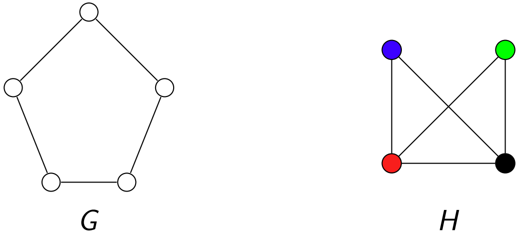

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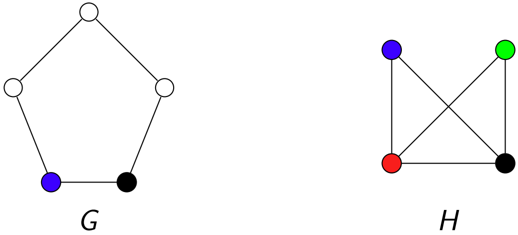

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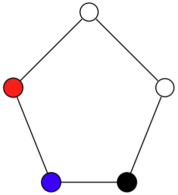

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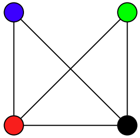

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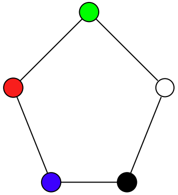

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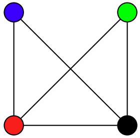

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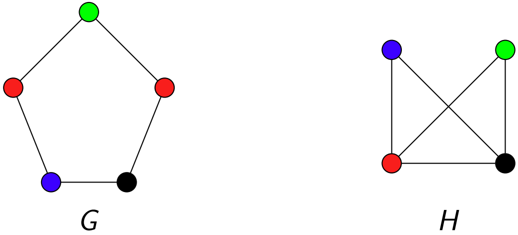

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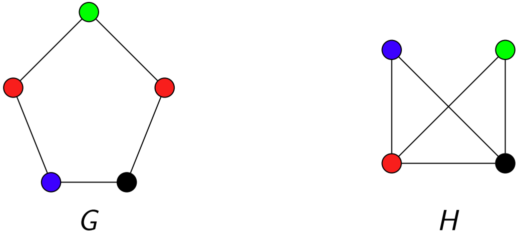

- A homomorphism $G \to H$ is a function $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$.
- If $H \cong K_n$, then a homomorphism $G \to H$ is an n-colouring of G.

Locally injective homomorphisms

A homomorphism $G \to H$ is locally injective if its restriction to $N(x)$ is injective, for every $x \in V(G)$.

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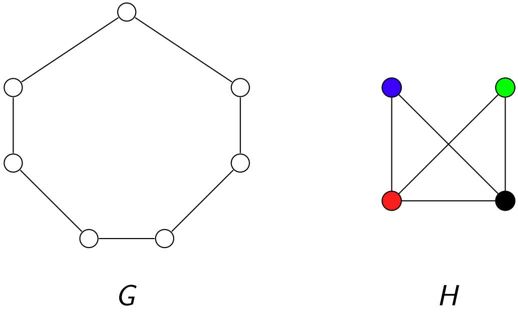

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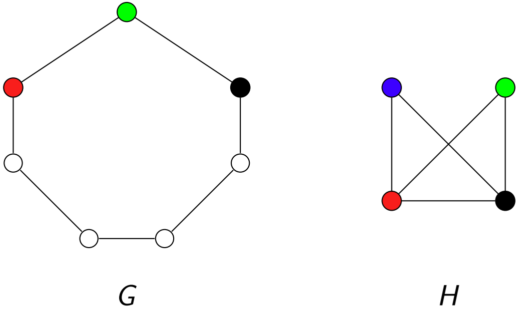

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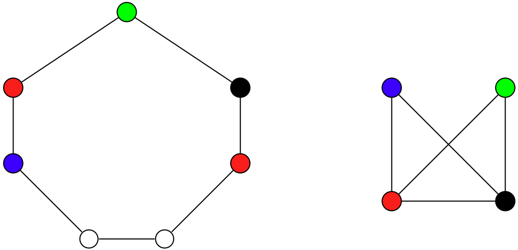

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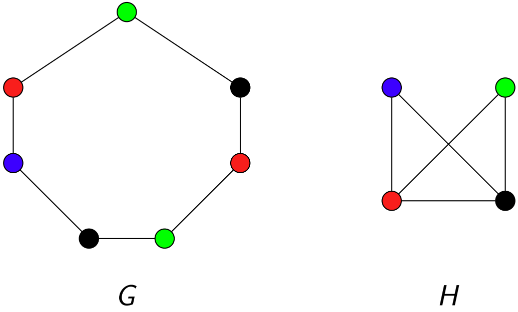

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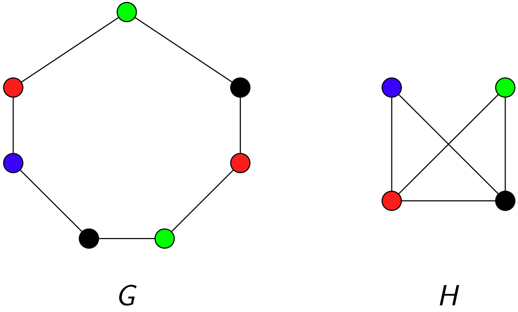
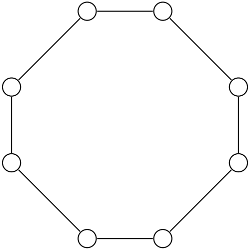


When $H \cong K_n$, a locally injective homomorphism $G \to H$ is a locally injective proper n-colouring.

Locally injective proper n-colourings: I

Properly colours the vertices so that no two vertices with a common neighbour get the same colour (the colouring is injective on neighbourhoods).



- Colourings of the square.

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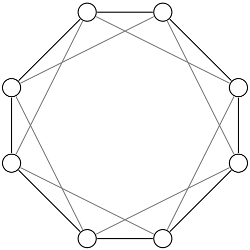

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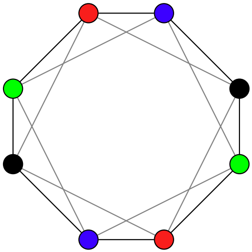

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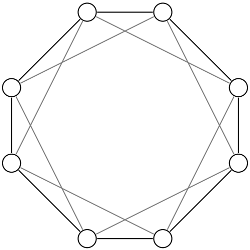

- Colourings of the square.
- $\Delta + 1$ colours needed; $\Delta^2 + 1$ colours suffice.

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  - There is a dichotomy in the list version [Fiala & Kratochvíl, 2006].

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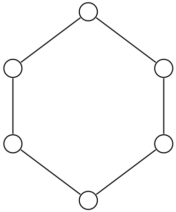
- A graph is reflexive if it has a loop at every vertex.
- If H is reflexive and $G \to H$ is a homomorphism, adjacent vertices of G can have the same "colour" (image), even in an injective homomorphism.



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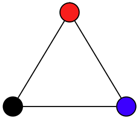

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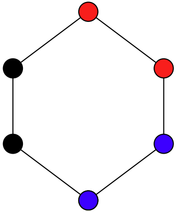

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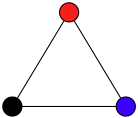

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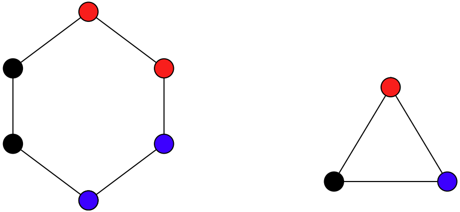

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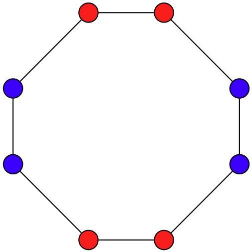

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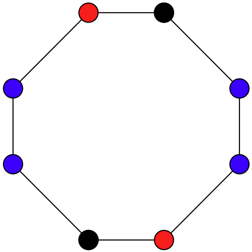

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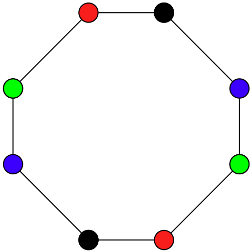

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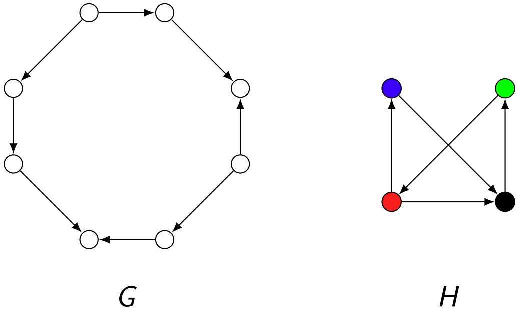

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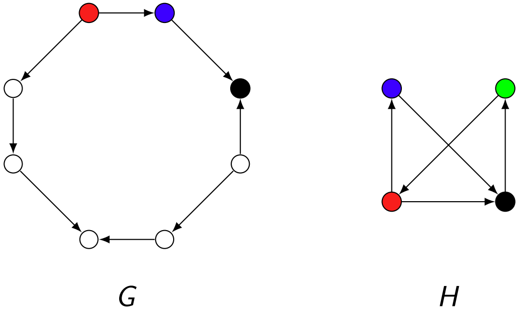

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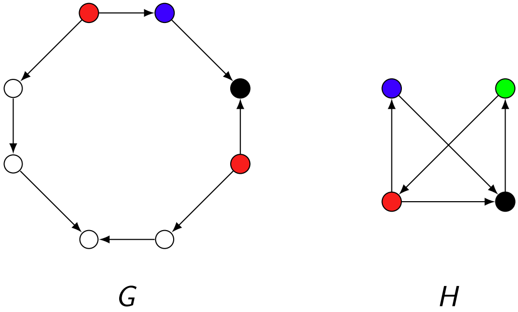

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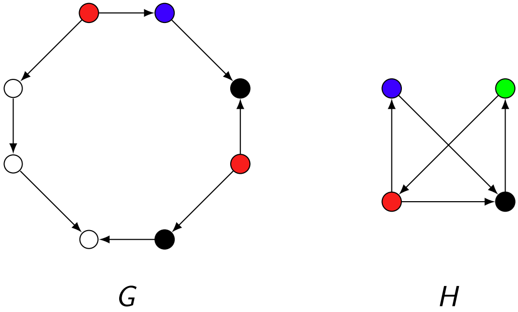

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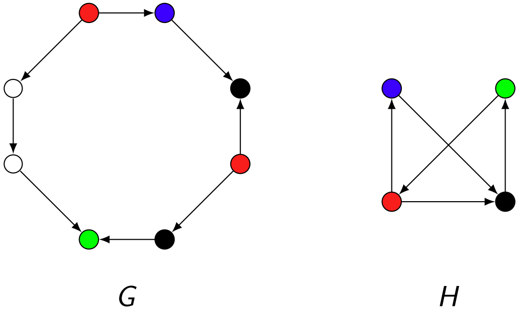

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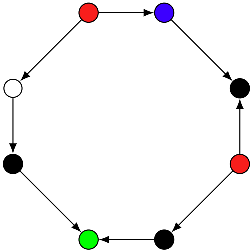

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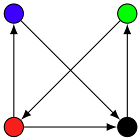

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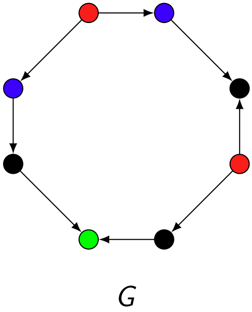

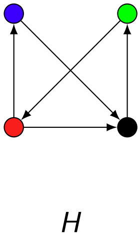

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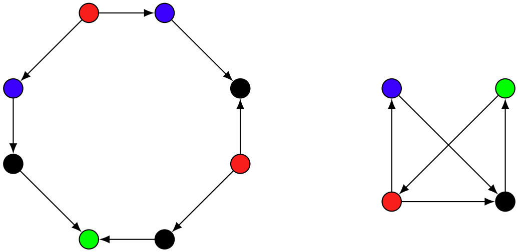

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What should injective mean?

Injective homomorphisms: the options

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1. is the most extensively studied [Swarts, 2008].

- When the target oriented graph H is reflexive, there is a dichotomy.
- When the target oriented graph is irreflexive (no loops), the complexity is at least as rich as for all digraph homomorphism problems.

Injective homomorphisms: the options

There are 3 possible definitions of injective for homomorphisms of oriented graphs.

1. injective on in-neighbourhoods only
2. injective on in- and out-neighbourhoods separately
3. injective on in- and out-neighbourhoods together

2. and 3. We know the complexity for all tournaments on small numbers of vertices, and in several infinite familiies. In both cases

- When the target tournament H is reflexive, the problem is polynomial if $|V(H)| \leq 2$, and NP-complete if $|V(H)| = 3$.
- When the target tournament H is irreflexive, the problem is polynomial if $|V(H)| \leq 3$, and NP-complete if $|V(H)| = 4$.

[Campbell, Clarke, & GM, 2011]

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- A description of the oriented graphs that are colourable can be obtained in the Polynomial cases (a touch ugly).

Summary ... a.k.a. the last slide

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- `Thank you for listening, reading, and not throwing tomatoes.`