The Width of "Canonical" Trees and of Acyclic Digraphs

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Partitions of 1 into Powers of t

- integer base $t \geq 2$

Definition

$\mathcal{C}_{Partition}$ is the set of integer tuples $(x_1, \ldots, x_r)$ for which

- $0 \leq x_1 \leq x_2 \leq \cdots \leq x_r$,
- $1 = \sum_{i=1}^{r} \frac{1}{t^{x_i}}$.

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E.g. $t = 2$, partitions with 5 summands:

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}. \end{aligned}$$

Canonical Compact t-ary Prefix Codes

- integer $t \geq 2$

Lemma (Kraft–McMillan inequality)

$$\sum_{c \in C} \frac{1}{t^{\mathrm{length}(c)}} \leq 1.$$

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$\mathcal{C}_{Code}$ is the set of prefix codes $C \subseteq \{0, \ldots, t-1\}^*$ for which

- C compact (equality in Kraft–McMillan inequality)
- C canonical (lexicographic ordering of its words corresponds to a non-decreasing ordering of the word lengths)

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E.g. $t = 2$, codes of size 5:

$$\{0, 10, 110, 1110, 1111\}, \{0, 100, 101, 110, 111\}, \{00, 01, 10, 110, 111\}.$$

Canonical Trees

- integer $t \geq 2$

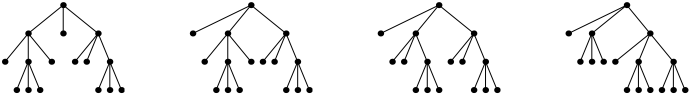
Definition (Rooted t-ary Tree)

- one vertex has been designated as the root
- all vertices have t (“inner vertex”) or no (“leaf”) successors

Definition (Canonical Rooted t-ary Tree)

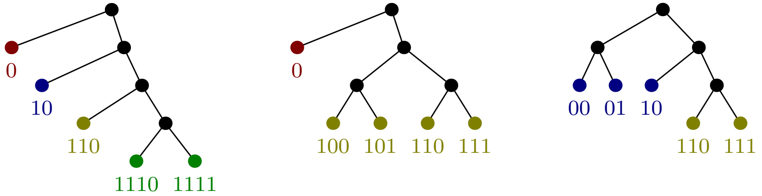
- the longer paths are as far to the right hand side as possible

(also called level-greedy tree)



Only the last tree is canonical.

Partitions, Codes and Trees



$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \qquad 1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \qquad 1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$$

Identification

$$\mathcal{C}_{Partition} = \mathcal{C}_{Code} = \mathcal{C}_{Tree}$$

Number of Canonical Trees

- asymptotic formula (main term)

$$\sim R\rho^n$$

with $\rho \to 2$ for increasing t

(Boyd 1975, Komlos–Moser–Nemetz 1984, Flajolet–Prodinger 1987)

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(Boyd 1975, Komlos–Moser–Nemetz 1984, Flajolet–Prodinger 1987)

- more precisely

$$R\rho^n + R_2\rho_2^n + O(r_3^n)$$

with

  - $\rho > \rho_2 > r_3$, R, R_2 positive real constants depending on t
  - asymptotic expansions (as $t \to \infty$)
    - $\rho = 2 - \frac{1}{2^{t+1}} + O\left(\frac{t}{2^{2t}}\right)$
    - $R = \frac{1}{8} + \frac{t-2}{2^{t+5}} + O\left(\frac{t^2}{2^{2t}}\right)$
    - $\rho_2 = \dots,\ R_2 = \dots,\ r_3 = \dots,$
  - explicit O-constants

(Elsholtz–Heuberger–Prodinger 2012)

Some Parameters of Canonical Trees

Theorem (Heuberger–K–Wagner 2012)

- *height*
  - *asymptotically normally distributed*
  - *mean* $\sim \mu_h n$
  - *variance* $\sim \sigma_h^2 n$
- *number of distinct summands*
  - *asymptotically normally distributed*
  - *mean* $\sim \mu_d n$
  - *variance* $\sim \sigma_d^2 n$
- *total path length*
  - *asymptotically normally distributed*
  - *mean* $\sim \mu_{tpl} n^2$
  - *variance* $\sim \sigma_{tpl}^2 n^3$
- *maximum number of equal summands*
  - *mean* $\mu_w \log n + O(\log \log n)$
  - *concentration property*
- *number of leaves on the last level*
  - *discrete limit distribution*
  - *mean* $2t + o(1)$
  - *variance* $2t^2 + o(1)$

Width of Canonical Trees

Theorem (Heuberger–K–Wagner 2012)

- *random canonical tree T of size n*
- *width has expectation*

$$\mathbb{E}(w(T)) = \mu_w \log n + O(\log\log n)$$

with

$$\mu_w = \frac{1}{-(t-1)\log q_0} = \frac{1}{t\log(2)} + \frac{1}{t^2\log(2)} + O\left(\frac{1}{t^2}\right)$$

- *concentration property*

$$\mathbb{P}(|w(T) - \mu_w \log n| \geq 3\mu_w \log\log n) = O\left(\frac{1}{\log n}\right)$$

Labelled Acyclic Digraphs

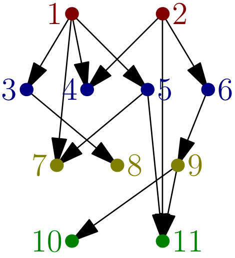

Figure: An acyclic digraph with height 3 and width 4.

- representation with "layers"
- height ⇝ number of layers (length of a longest path)
-
  - top layer: sources
  - bottom layer: sinks
- width is maximal number of vertices at one layer

Some Properties

- number of labelled acyclic digraphs
  - asymptotic formula

    $$\sim C\rho^n 2^{\binom{n}{2}} n!$$

    with positive constants C, ρ

  (Robinson 1973, Bender–Richmond–Robinson–Wormald 1986)

Some Properties

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  (Robinson 1973, Bender–Richmond–Robinson–Wormald 1986)
- height of an acyclic digraph
  - asymptotically normally distributed
  - mean $\sim \mu_h n$
  - variance $\sim \sigma_h^2 n$

  (McKay 1989)

Width of Acyclic Digraphs

Theorem (K–Wagner 2013)

- *random (labelled) acyclic digraph D of size n*
- *width has expectation*

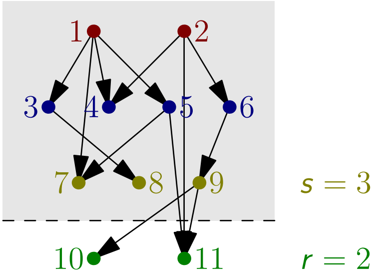
$$\mathbb{E}(w(D)) = \sqrt{\log_S n} + O(\log\log n)$$

with $S = \sqrt{2}$

- *concentration property*

$$\mathbb{P}\left(\left|w(D) - \sqrt{\log_S n}\right| \geq \log_S \log_S n\right) = O\left(\frac{1}{\sqrt{\log_S n}^{3\sqrt{\log_S n}}}\right)$$

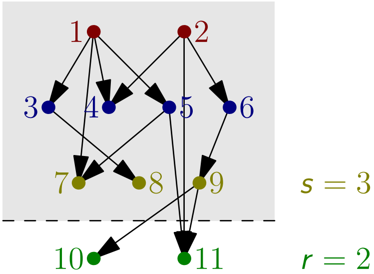
Infinite Transfermatrix



- building an acyclic digraph
  - with r sinks
  - out of a one with s sinks
  - by adding a layer

$$M_\infty = \begin{pmatrix} & \cdots & \\ \cdots & \frac{q^r}{2^{\binom{r}{2}} r!} (1 - 2^{-s})^r & \cdots \\ & \cdots & \end{pmatrix}$$

Infinite Transfermatrix



- building an acyclic digraph
  - with r sinks
  - out of a one with s sinks
  - by adding a layer
- leads to recursion
- generating function

$$G(q) = \sum_{\substack{D \text{ acyclic digraph} \\ n = \text{size of } D}} \frac{q^n}{2^{\binom{n}{2}} n!}$$

equals

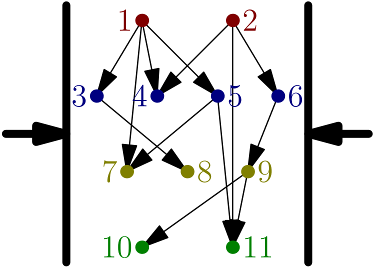
$$\sum_{\text{all matrix entries}} (I - M_\infty)^{-1} S$$

$$M_\infty = \begin{pmatrix} & \cdots & \\ \cdots & \frac{q^r}{2^{\binom{r}{2}} r!} (1 - 2^{-s})^r & \cdots \\ & \cdots & \end{pmatrix}$$

Width-Restrictions

- restrict to acyclic digraphs with width $\leq W$
- $\leadsto$ finite transfer matrix

$$M_W = \begin{pmatrix} & \cdots & \\ \cdots & \frac{q^r}{2^{\binom{r}{2}} r!} (1 - 2^{-s})^r & \cdots \\ & \cdots & \end{pmatrix}_{\substack{r \leq W \\ s \leq W}}$$



- $\leadsto$ generating function

$$G_W(q) = \sum_{\substack{D \text{ acyclic digraph} \\ \text{with width} \leq W \\ n = \text{size of } D}} \frac{q^n}{2^{\binom{n}{2}} n!} = \sum_{\text{all matrix entries}} (I - M_W)^{-1} S$$

Main Steps in the Proof

- generating function $G(q)$, $G_W(q)$
- dominant singularities q_∞, q_W
- singularity analysis

$$\mathbb{P}(\text{width of acyclic digraph} \leq W) \sim c_W \left(\frac{q_W}{q_\infty}\right)^{-n-1}$$

with

- $c_W \to 1$ (comes from finite/infinite determinants)

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- expectation

$$\mathbb{E}(\text{width of acyclic digraph}) = \sum_{W \geq 0} (1 - \mathbb{P}(\text{width} \leq W))$$

- splitting up the sum into

Dominant Singularities

- generating function

$$G(q) = \sum_{\text{all matrix entries}} (I - M_\infty)^{-1} S$$

with (infinite) transfer matrix M_∞

- dominant singularity $\leftrightsquigarrow$ eigenvalue 1 of M_∞

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- dominant singularity ⇜⇝ eigenvalue 1 of M_∞
- infinite positive eigenvector
- truncate and apply methods from Perron–Frobenius theory

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  with (infinite) transfer matrix M_∞
- dominant singularity $\leftrightsquigarrow$ eigenvalue 1 of M_∞
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- truncate and apply methods from Perron–Frobenius theory
- q_W converges to q_∞ with rate $\sqrt{2}^{-W^2}$

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