Eulerain and Stirling numbers over multisets

Joon Yop Lee

POSTECH

Eulerian numbers I

Let $\left\langle {d \atop i} \right\rangle$ be an *Eulerian number*, which is the number of permutations of $\{1, 2, \ldots, d\}$ with i descents.

Example

Let $\sigma = [1, 3, 2, 5, 4]$ be a permutation of $\{1, 2, 3, 4, 5\}$. The number of descent in σ is 2, that is, $3 > 2$ and $5 > 4$.

Eulerian numbers II

Worpitzky identity

$$x^d = \sum_{i=0}^{d-1} \left\langle {d \atop i} \right\rangle \binom{x-1-i+d}{d},$$

Carlitz identity (A q-analog of Worpitzky identity)

$$\begin{bmatrix} x \\ 1 \end{bmatrix}_q^d = \sum_{i=1}^{d} A_{d,i}(q) \begin{bmatrix} x-1+i \\ d \end{bmatrix}_q$$

where $\left[{x \atop m}\right]_q = \prod_{i=1}^{m}(1-q^{x-m+i})/(1-q^i)$ and $A_{d,i}(q)$ is a polynomial of q.

Observation

- Ordinary sets $\{1, 2, \ldots, l\} \to$ multisets $\{1^{d_1}, 2^{d_2}, \ldots, l^{d_l}\}$.
- Eulerian numbers $\to$ multiset Eulerian numbers?
- Worpitzky identity $\to$ a multiset version?.
- Carlitz identity $\to$ a multiset version?.

Stirling numbers of the second kind

Let $\left\{ {d \atop k} \right\}$ be a *Stirling number of the second kind*, which is the number of partitions of a d element set into k nonempty sets.

Stirling identity

$$x^d = \sum_{k=1}^{d} k! \left\{ {d \atop k} \right\} \binom{x}{k}$$

We call $k!\left\{ {d \atop k} \right\}$ an *ordered Stirling numbers of the second kind*.

Observation

- Ordered Stirling numbers $\rightarrow$ multiset ordered Stirling numbers?
- Stirling identity $\rightarrow$ a multiset version? or a q-analog?

Goals

- Multiset versions of Worpitzky identity and Carlitz ideneity.
- A multiset version of Stirling identity and its q-analog.
- Computations of multiset Eulerian numbers and multiset ordered Stirling numbers of the second kind.

Basic ideas

- For a sequence of finite sets of lattice points $S_0, S_1, S_2, \ldots,$ we compute the numbers of elements in S_n by two different ways and obtain a polynomial identity.
- To obtain a q-analog of this identity, we compute the following generating function

$$\sum_{(x_1,x_2,\ldots,x_d)\in S_n} q^{x_1+x_2+\cdots+x_d}$$

by two different ways.

Notations I

- For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$.
- We denote point in $\mathbb{R}^d$ by $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and denote $q^{\mathbf{x}} = q^{x_1+x_2+\cdots+x_d}$.
- The zero vector of $\mathbb{R}^d$ is denoted by $\mathbf{e}_{d,0}$ and for each $1 \leq i \leq d$ the ith unit vector of $\mathbb{R}^d$ is denoted by $\mathbf{e}_{d,i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 in the ith coordinate).

Notations II

- We define α_0^d to be a d-dimensional simplex whose vertexes are $\mathbf{e}_{d,0}$, $\mathbf{e}_{d,0}+\mathbf{e}_{d,1}, \ldots,$ $\mathbf{e}_{d,0}+\mathbf{e}_{d,1}+\cdots+\mathbf{e}_{d,d}$. Note that α_0^d is the set of points $\mathbf{x}$ such that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$.
- Let $S(\mathbf{d}) = \{1^{d_1}, 2^{d_2}, \ldots, l^{d_l}\}$ be a multiset such that for each $1 \leq j \leq l$ the number of j in $S(\mathbf{d})$ is d_j. We define $\mathfrak{S}(\mathbf{d})$ to be the permutation set of $S(\mathbf{d})$ and denote a permutation of $S(\mathbf{d})$ by $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_d]$.

The product of simplexes I

- Let $d = d_1 + d_2 + \cdots + d_l$ be a sum of nonnegative integers. For each $1 \leq j \leq l$, writing a point of $\mathbb{R}^{d_j}$ by $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,d_j})$, we also denote a point of $\mathbb{R}^d = \prod_{j=1}^{l} \mathbb{R}^{d_j}$ by $\mathbf{x} = \prod_{j=1}^{l} \mathbf{x}_j$.
- Let $\mathbf{d} = (d_1, d_2, \ldots, d_l)$. We define $\alpha^{\mathbf{d}} = \prod_{j=1}^{l} \alpha_0^{d_j}$ and denote the vertexes of $\alpha^{\mathbf{d}}$ by $\mathbf{e}_{i_1,i_2,\ldots,i_l} = \prod_{j=1}^{l}(\mathbf{e}_{d_j,0} + \mathbf{e}_{d_j,1} + \cdots + \mathbf{e}_{d_j,i_j})$ for $(i_1, i_2, \ldots, i_l) \in \prod_{j=1}^{l}\{0, 1, \ldots, d_j\}$.

A triangulation of the product of simplexes I

- For a permutation σ of $S(\mathbf{d})$, we define $\alpha^d(\sigma)$ to be a d-simplex with the vertexes $\mathbf{e}_{i_{h,1},i_{h,2},\ldots,i_{h,l}}$ for $0 \leq h \leq d$ such that

$$\begin{cases} (i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = \mathbf{e}_{l,0}, \\ (i_{h,1}, i_{h,2}, \ldots, i_{h,l}) = \sum_{a=1}^{h} \mathbf{e}_{l,\sigma_a} \text{ for } 1 \leq h \leq d. \end{cases}$$

A triangulation of the product of simplexes II

- We can easily show that

$$\alpha^{\mathbf{d}} = \bigcup_{\sigma \in \mathfrak{S}(\mathbf{d})} \alpha^{d}(\sigma).$$

- Moreover, the set $\mathcal{T}_{\alpha^{\mathbf{d}}}$ which is composed of the d-simplexes $\alpha^{d}(\sigma)$ for $\sigma \in \mathfrak{S}(\mathbf{d})$ and their faces is a triangulation of $\alpha^{\mathbf{d}}$.

Main idea

- Let R be a subset of $\mathbb{R}^d$ and n be a nonnegative integer. We denote $nR = \{nr \mid r \in R\}$ and $\mathbb{Z}(R) = \{\mathbf{x} \in R \mid \mathbf{x} \in \mathbb{N}^d\}$.
- To obtain a multiset version of Worpitzky identity, we will compute $|\mathbb{Z}(n\alpha^{\mathbf{d}})|$ by two different ways.
- To obtain a multiset version of Carlitz identity, we will compute $f_1(q) = \sum_{\mathbf{x} \in \mathbb{Z}(n\alpha^{\mathbf{d}})} q^{\mathbf{x}}$ by two different ways.

The first decomposition of $\alpha^{\mathbf{d}}$

- For a permutation σ of $S(\mathbf{d})$, let $D(\sigma)$ be the *descent set* of σ, that is, the set of indexes h such that $\sigma_h > \sigma_{h+1}$. We define

$$A(\sigma) = \{\mathbf{x} \in \alpha^{d}(\sigma) \mid x_{\sigma_h, i_h} > x_{\sigma_{h+1}, i_{h+1}} \text{ for } h \in D(\sigma)\}.$$

- We define the *first decomposition* of $\alpha^{\mathbf{d}}$ to be

$$\alpha^{\mathbf{d}} = \biguplus_{\sigma \in \mathfrak{S}(\mathbf{d})} A(\sigma)$$

where $\biguplus$ is the disjoint union.

A multiset version of Worpitzky identity I

- By definition,

$$|\mathbb{Z}(n\alpha^{\mathbf{d}})| = \prod_{j=1}^{l} |\mathbb{Z}(n\alpha^{d_j})| = \prod_{j=1}^{l} \binom{n+d_j}{d_j}.$$

- $\mathbf{x} \in \mathbb{Z}(nA(\sigma))$ if and only $\mathbf{x}$ is a lattice point such that
  1. $n \geq x_{\sigma_1,i_1} \geq x_{\sigma_2,i_2} \geq \cdots \geq x_{\sigma_d,i_d} \geq 0$,
  2. $x_{\sigma_h,i_h} \geq x_{\sigma_{h+1},i_{h+1}} + 1$ for $h \in D(\sigma)$.

  Thus

$$|\mathbb{Z}(nA(\sigma))| = \binom{n-|D(\sigma)|+d}{d}.$$

A multiset version of Worpitzky identity II

- Therefore if we denote by $\left\langle {\mathbf{d} \atop i} \right\rangle$ the number of permutations of $S(\mathbf{d})$ with i descents, called a *multiset Eulerian number*, then by the first decomposition of $\alpha^{\mathbf{d}}$ we obtain

$$|\mathbb{Z}(n\alpha^{\mathbf{d}})| = |\mathbb{Z}(\biguplus_{\sigma\in\mathfrak{S}(\mathbf{d})} nA(\sigma))| = \sum_{\sigma\in\mathfrak{S}(\mathbf{d})} |\mathbb{Z}(nA(\sigma))|$$

$$= \sum_{\sigma\in\mathfrak{S}(\mathbf{d})} \binom{n-|D(\sigma)|+d}{d} = \sum_{i=1}^{d-1} \left\langle {\mathbf{d} \atop i} \right\rangle \binom{n-i+d}{d}.$$

A multiset version of Worpitzky identity III

- As a result,

$$\prod_{j=1}^{l}\binom{n+d_j}{d_j}=\sum_{i=0}^{d-1}\left\langle {\mathbf{d} \atop i} \right\rangle\binom{n-i+d}{d}.$$

A multiset version of Wopritzky identity

$$\prod_{j=1}^{l}\binom{x+d_j}{d_j}=\sum_{i=0}^{d-1}\left\langle {\mathbf{d} \atop i} \right\rangle\binom{x-i+d}{d}.$$

Computations of multiset Eulerian numbers I

- From identity (3), we can obtain the following matrix identity

$$\begin{bmatrix} \prod_{j=1}^{l}\binom{d_j}{d_j} \\ \prod_{j=1}^{l}\binom{1+d_j}{d_j} \\ \cdots \\ \prod_{j=1}^{l}\binom{d-1+d_j}{d_j} \end{bmatrix} = \begin{bmatrix} \binom{d}{d} & 0 & \cdots & 0 \\ \binom{d+1}{d} & \binom{d}{d} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & \binom{d}{d} \end{bmatrix} \begin{bmatrix} \left\langle {\mathbf{d} \atop 0} \right\rangle \\ \left\langle {\mathbf{d} \atop 1} \right\rangle \\ \cdots \\ \left\langle {\mathbf{d} \atop d-1} \right\rangle \end{bmatrix}.$$

- By using the Gaussin elimination, we obtain

Multiset Eulerain numbers

$$\left\langle {\mathbf{d} \atop i} \right\rangle = \sum_{h=0}^{i}(-1)^{i-h}\binom{d+1}{i-h}\prod_{j=1}^{l}\binom{h+d_j}{d_j}.$$

A multiset version of Carlitz identity I

- $\mathbf{x}_j \in \mathbb{Z}(n\alpha_0^{d_j})$ if and only if $n \geq x_{j,1} \geq x_{j,2} \geq \cdots \geq x_{j,d_j} \geq 0$.

  Thus

$$\sum_{\mathbf{x}_j \in \mathbb{Z}(n\alpha_0^{d_j})} q^{\mathbf{x}_j} = \begin{bmatrix} n+d_j \\ d_j \end{bmatrix}_q.$$

- From this result, we obtain

$$f_1(q) = \prod_{j=1}^{l} \sum_{\mathbf{x}_j \in \mathbb{Z}(n\alpha_0^{d_j})} q^{\mathbf{x}_j} = \prod_{j=1}^{l} \begin{bmatrix} n+d_j \\ d_j \end{bmatrix}_q.$$

A multiset version of Carlitz identity II

- For a permutation σ of $S(\mathbf{d})$, we define the *major index* of σ to be $maj(\sigma) = \sum_{j \in D(\sigma)} j$. A point $\mathbf{x}$ is in $\mathbb{Z}(nA(\sigma))$ if and only if

$$\begin{cases} x_{\sigma_h, i_h} \geq x_{\sigma_{h+1}, i_{h+1}} & \text{for } h \in [d+1] \setminus D(\sigma) \\ x_{\sigma_h, i_h} \geq x_{\sigma_{h+1}, i_{h+1}} + 1 & \text{for } h \in D(\sigma) \end{cases}.$$

Therefore

$$\sum_{\mathbf{x} \in \mathbb{Z}(nA(\sigma))} q^{\mathbf{x}} = q^{maj(\sigma)} \begin{bmatrix} n - |D(\sigma)| + d \\ d \end{bmatrix}_q.$$

A multiset version of Carlitz identity III

- Since $\mathbb{Z}(n\alpha^{\mathbf{d}}) = \biguplus_{\sigma\in\mathfrak{S}(\mathbf{d})} \mathbb{Z}(nA(\sigma))$, we obtain

$$
\begin{aligned}
f_1(q) &= \sum_{\sigma\in\mathfrak{S}(\mathbf{d})} \sum_{\mathbf{x}\in\mathbb{Z}(nA(\sigma))} q^{\mathbf{x}} \\
&= \sum_{\sigma\in\mathfrak{S}(\mathbf{d})} q^{maj(\sigma)} \begin{bmatrix} n-|D(\sigma)|+d \\ d \end{bmatrix}_q \\
&= \sum_{i=1}^{d} A_{\mathbf{d},i}(q) \begin{bmatrix} n+i \\ d \end{bmatrix}_q
\end{aligned}
$$

where $A_{\mathbf{d},i}(q) = \sum_{\substack{\sigma\in\mathfrak{S}(\mathbf{d}) \\ |D(\sigma)|=d-i}} q^{maj(\sigma)}$.

A multiset version of Carlitz identity IV

- As a result,

$$\prod_{j=1}^{l} \begin{bmatrix} n + d_j \\ d_j \end{bmatrix}_q = \sum_{i=1}^{d} A_{\mathbf{d},i}(q) \begin{bmatrix} n + i \\ d \end{bmatrix}_q .$$

A multiset version of Carlitz identity

$$\prod_{j=1}^{l} \begin{bmatrix} x + d_j \\ d_j \end{bmatrix}_q = \sum_{i=1}^{d} A_{\mathbf{d},i}(q) \begin{bmatrix} x + i \\ d \end{bmatrix}_q .$$

The second decomposition of $\alpha^{\mathbf{d}}$ I

- For two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_l)$ and $\mathbf{v}' = (v_1', v_2', \ldots, v_l')$, we define $\mathbf{v} \leq \mathbf{v}'$ if $v_i \leq v_i'$ for all $1 \leq i \leq l$. Let $\mathcal{I}(\alpha^{\mathbf{d}})$ be the set of simplexes in $\mathcal{T}_{\alpha^{\mathbf{d}}}$ that contains $\mathbf{e}_{0,0,\ldots,0}$ and $\mathbf{e}_{d_1,d_2,\ldots,d_l}$.
- Let α^k be a k-dimensional simplex in $\mathcal{I}(\alpha^{\mathbf{d}})$. Then the vertex set of α^k is of the form $\{\mathbf{e}_{i_{h,1},i_{h,2},\ldots,i_{h,l}} \mid 0 \leq h \leq k\}$ where

$$\begin{cases} (i_{0,1}, i_{0,2}, \ldots, i_{0,l}) = \mathbf{0} \\ (i_{h,1}, i_{h,2}, \ldots, i_{h,l}) < (i_{h+1,1}, i_{h+1,2}, \ldots, i_{h+1,l}) \text{ for } 0 \leq h \leq k-1 \\ (i_{k,1}, i_{k,2}, \ldots, i_{k,l}) = (d_1, d_2, \ldots, d_l). \end{cases} \tag{1}$$

The second decomposition of $\alpha^{\mathbf{d}}$ II

- Let $I(\alpha^k)$ be a subset of α^k which is composed of the set of convex sums $\mathbf{x} = \sum_{h=0}^{k} c_h \mathbf{e}_{i_{h,1}, i_{h,2}, \ldots, i_{h,l}}$ such that $c_h > 0$ for $1 \leq h \leq k-1$.
- We define the *second decomposition* of $\alpha^{\mathbf{d}}$ to be

$$\alpha^{\mathbf{d}} = \biguplus_{k=1}^{d} \biguplus_{\alpha^k \in \mathcal{I}(\alpha^{\mathbf{d}})} I(\alpha^k).$$

A multiset version of Stirling identity I

- Let α^k be a k-dimensional simplex in $\mathcal{I}(\alpha^{\mathbf{d}})$ with the vertex set $\{\mathbf{e}_{i_{h,1},i_{h,2},\ldots,i_{h,l}} \mid 0 \leq h \leq k\}$. For each $1 \leq h \leq k$ if we denote $S_h = S(i_{h,1}, i_{h,2}, \ldots, i_{h,l}) \setminus S(i_{h-1,1}, i_{h-1,2}, \ldots, i_{h-1,l})$, then $(S_1, S_2, \ldots, S_k)$ is an ordered partition of $S(\mathbf{d})$ into k nonempty multisets.

A multiset version of Stirling identity II

- Therefore the number of k-dimensional simplexes in $\mathcal{I}(\alpha^{\mathbf{d}})$ is the number of ordered partitions of $S(\mathbf{d})$ into k nonempty multisets. We call this number an *ordered multiset Stirling number of the second kind* and denote it by $\left\{ {\mathbf{d} \atop k} \right\}_O$. Note that if $d_1 = d_2 = \cdots = d_l = 1$, then $\left\{ {\mathbf{d} \atop k} \right\}_O = k! \left\{ {l \atop k} \right\}$.

A multiset version of Stirling identity III

- The number of elements in the set $\mathbb{Z}(nI(\alpha^k))$ is $\binom{n+1}{k}$. Thus by the second decomposition of $\alpha^{\mathbf{d}}$ it follows that

$$|\mathbb{Z}(n\alpha^{\mathbf{d}})| = |\mathbb{Z}(n \biguplus_{k=1}^{d} \biguplus_{\alpha^k \in \mathcal{I}(\alpha^{d_1,\dots,d_l})} I(\alpha^k))|$$

$$= \sum_{k=1}^{d} \sum_{\alpha^k \in \mathcal{I}(\alpha^{\mathbf{d}})} |\mathbb{Z}(nI(\alpha^k))|$$

$$= \sum_{k=1}^{d} \left\{ \begin{matrix} \mathbf{d} \\ k \end{matrix} \right\}_O \binom{n+1}{k}.$$

A multiset version of Stirling identity IV

- As a result,

$$\prod_{j=1}^{l} \binom{n+d_j}{d_j} = \sum_{k=1}^{d} \left\{ \begin{matrix} \mathbf{d} \\ k \end{matrix} \right\}_O \binom{n+1}{k}.$$

A multiset version of Stirling identity

$$\prod_{j=1}^{l} \binom{x+d_j}{d_j} = \sum_{k=1}^{d} \left\{ \begin{matrix} \mathbf{d} \\ k \end{matrix} \right\}_O \binom{x+1}{k}.$$

Computations of multiset Stirling numbers of the second kind I

- Multiset ordered Stirling numbers of the second kind satisfy

$$\begin{bmatrix} \prod_{j=1}^{l}\binom{d_j}{d_j} \\ \prod_{j=1}^{l}\binom{1+d_j}{d_j} \\ \cdots \\ \prod_{j=1}^{l}\binom{d-1+d_j}{d_j} \end{bmatrix} = \begin{bmatrix} \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \binom{d}{1} & \binom{d}{2} & \cdots & \binom{d}{d} \end{bmatrix} \begin{bmatrix} \left\{ \mathbf{d} \atop 1 \right\}_O \\ \left\{ \mathbf{d} \atop 2 \right\}_O \\ \cdots \\ \left\{ \mathbf{d} \atop d \right\}_O \end{bmatrix}.$$

Computations of multiset Stirling numbers of the second kind II

- By the Gaussian elimination, we obtain

Multiset ordered Stirling numbers of the second

$$\left\{ \begin{matrix} \mathbf{d} \\ k \end{matrix} \right\}_O = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k}{h} \prod_{j=1}^{l} \binom{h+d_j}{d_j}.$$

A q-analog of a multiset version of Stirling identity I

- Let α^k be a k-dimensional simplex in $\mathcal{I}(\alpha^{\mathbf{d}})$ with the vertex set $\{\mathbf{e}_{i_{h,1},i_{h,2},\ldots,i_{h,l}} \mid 0 \le h \le k\}$. We define the *major index* of α^k to be

$$maj(\alpha^k) = \sum_{h=1}^{k-1}\sum_{j=1}^{l} i_{h,j}.$$

A q-analog of a multiset version of Stirling identity II

- Each α^k in $\mathcal{I}(\alpha^{\mathbf{d}})$ satisfies $\sum_{\mathbf{x}\in\mathbb{Z}(nI(\alpha^k))} q^{\mathbf{x}} = q^{maj(\alpha^k)} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$.

  Thus it follows that

$$
\begin{aligned}
f_1(q) &= \sum_{k=1}^{d} \sum_{\alpha^k\in\mathcal{I}(\alpha^{\mathbf{d}})} \sum_{\mathbf{x}\in\mathbb{Z}(nI(\alpha^k))} q^{\mathbf{x}} \\
&= \sum_{k=1}^{d} \sum_{\alpha^k\in\mathcal{I}(\alpha^{\mathbf{d}})} q^{maj(\alpha^k)} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \\
&= \sum_{k=1}^{d} B_{\mathbf{d},k}(q) \begin{bmatrix} n+1 \\ k \end{bmatrix}_q
\end{aligned}
$$

  where $B_{\mathbf{d},k}(q) = \sum_{\alpha^k\in\mathcal{I}(\alpha^{\mathbf{d}})} q^{maj(\alpha^k)}$.

A q-analog of a multiset version of Stirling identity III

- As a result,

$$\prod_{j=1}^{l} \begin{bmatrix} n+d_j \\ d_j \end{bmatrix}_q = \sum_{k=1}^{d} B_{\mathbf{d},k}(q) \begin{bmatrix} n+1 \\ k \end{bmatrix}_q .$$

A q-analog of a multiset version of Stirling identity

$$\prod_{j=1}^{l} \begin{bmatrix} x+d_j \\ d_j \end{bmatrix}_q = \sum_{k=1}^{d} B_{\mathbf{d},k}(q) \begin{bmatrix} x+1 \\ k \end{bmatrix}_q .$$