Extremal Hypergraphs for

Packing and Covering

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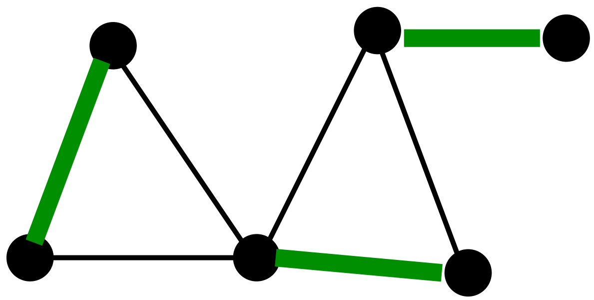
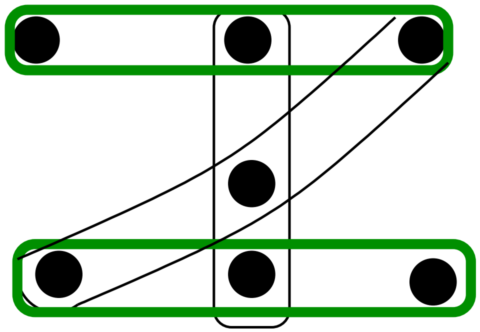
Joint work with L. Narins and T. Szabó

Packing

Let $\mathcal{H}$ be a hypergraph. A packing or matching of $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$.

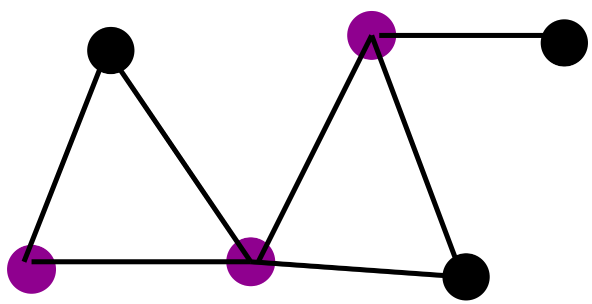


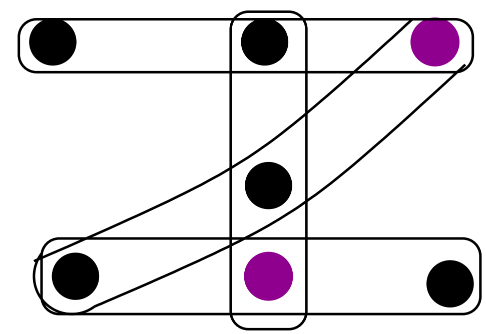


The parameter $\nu(\mathcal{H})$ is defined to be the maximum size of a packing in $\mathcal{H}$.

Covering

A cover of the hypergraph $\mathcal{H}$ is a set of vertices C of $\mathcal{H}$ such that every edge of $\mathcal{H}$ contains a vertex of C.

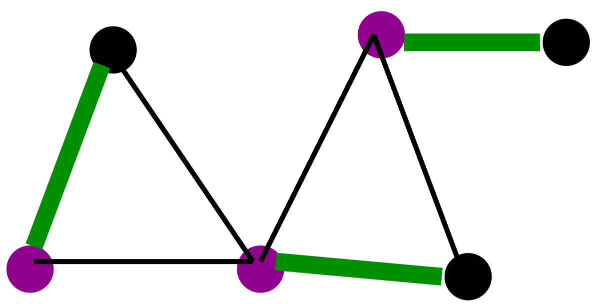


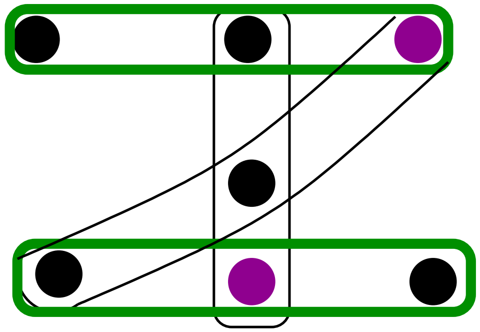


The parameter $\tau(\mathcal{H})$ is defined to be the minimum size of a cover of $\mathcal{H}$.

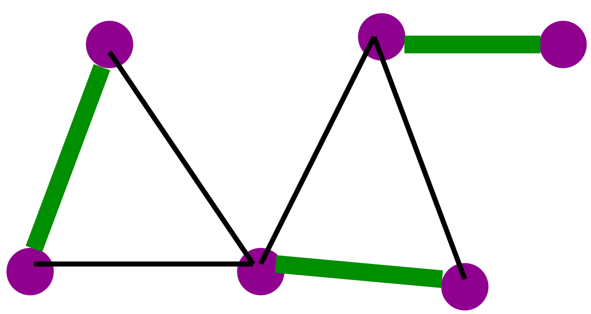
Comparing $\nu(\mathcal{H})$ and $\tau(\mathcal{H})$

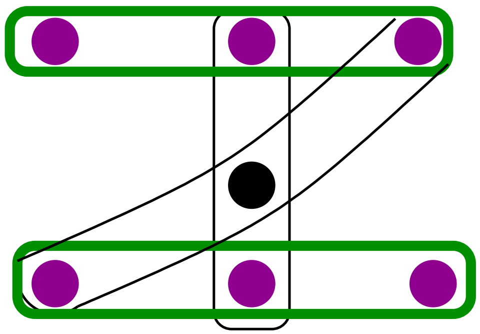
For every hypergraph $\mathcal{H}$ we have $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$.



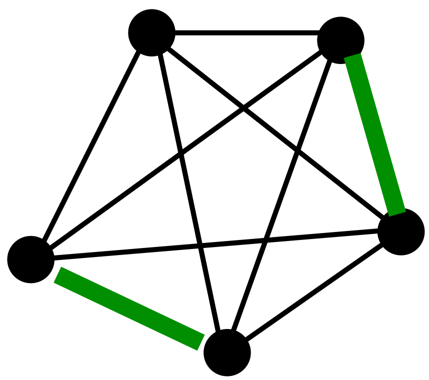


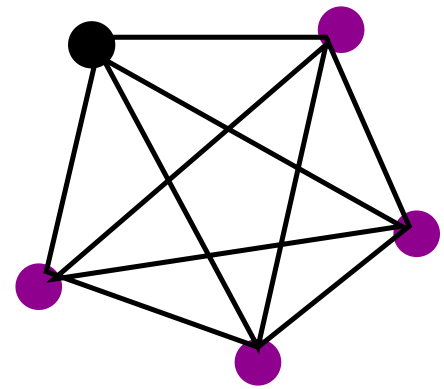
For every r-uniform hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$.





The upper bound $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$ is attained for certain hypergraphs, for example for the complete r-uniform hypergraph $\mathcal{K}^r_{rt+r-1}$ with $rt+r-1$ vertices, in which $\nu = t$ and $\tau = rt$.

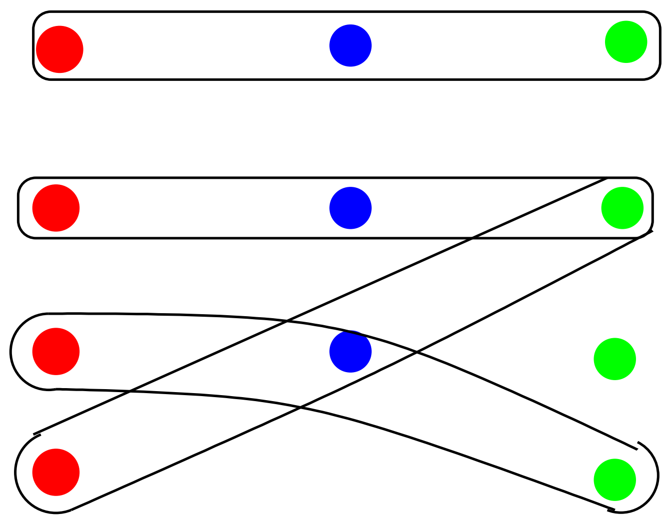




Ryser's Conjecture

Conjecture: Let $\mathcal{H}$ be an r-partite r-uniform hypergraph. Then

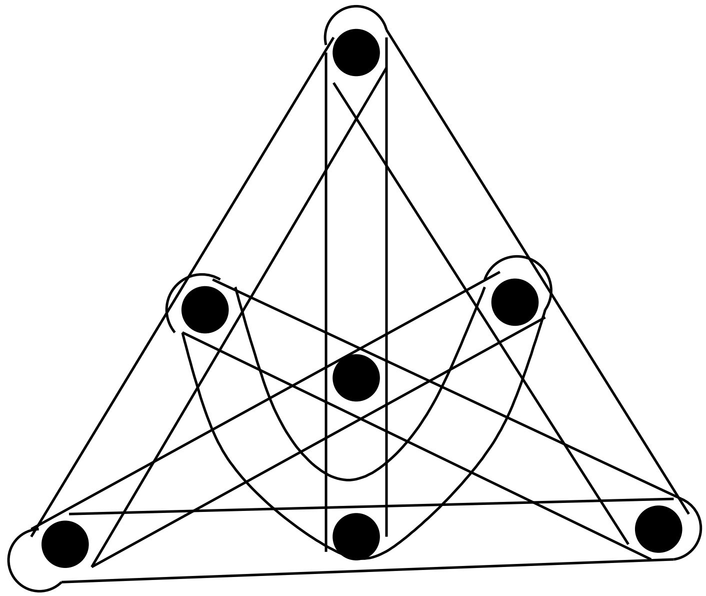
$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H}).$$

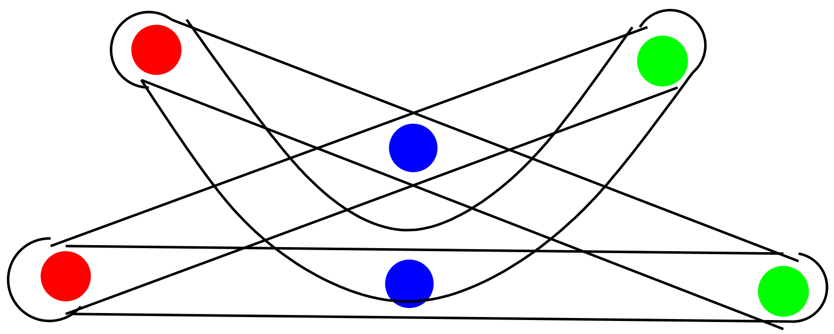


This conjecture dates from the early 1970's.

Results on Ryser's Conjecture

- $r = 2$: This is König's Theorem for bipartite graphs.
- $r = 3$: Known (proved by Aharoni, 2001)
- $r = 4$ and $r = 5$: Known for small values of $\nu(\mathcal{H})$, namely for $\nu(\mathcal{H}) \leq 2$ when $r = 4$ and for $\nu(\mathcal{H}) = 1$ when $r = 5$. (Tuza)
- whenever $r - 1$ is a prime power: If true, the upper bound is best possible.





Here $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = r - 1$.

On Ryser's Conjecture for $r = 3$

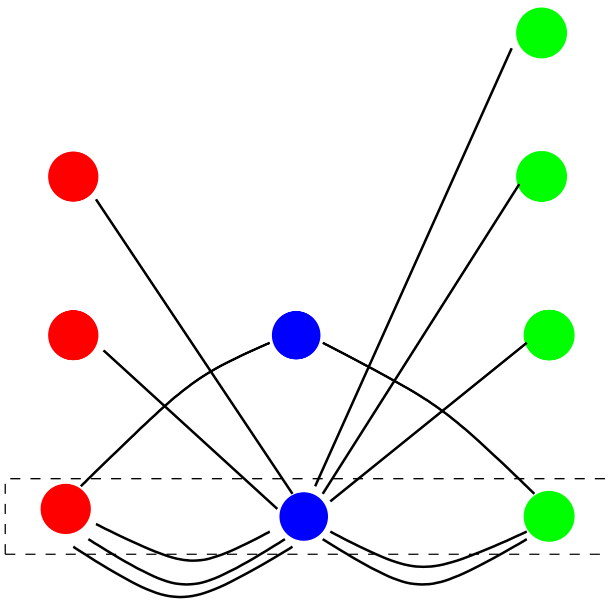
Theorem (Aharoni 2001): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then

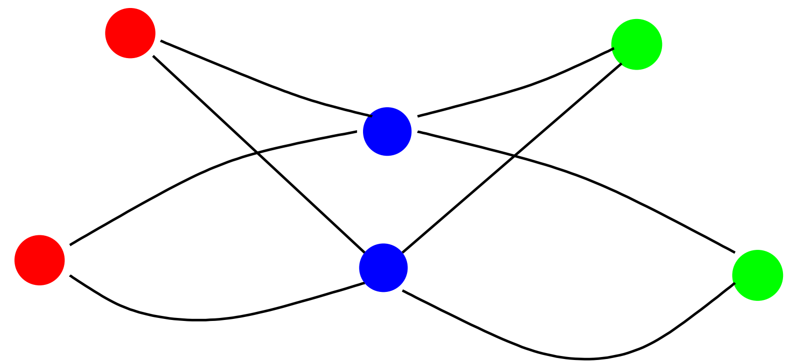
$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

Proof: Uses topological connectedness of matching complexes of bipartite graphs.

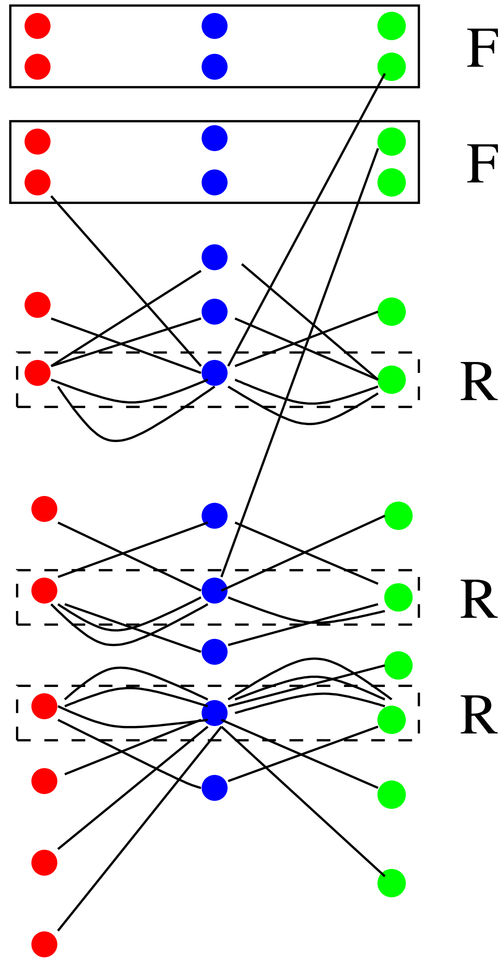
Q: What is $\mathcal{H}$ like if it is a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$?

Extremal hypergraphs for Ryser's Conjecture





Home base hypergraphs



Extremal hypergraphs for Ryser's Conjecture

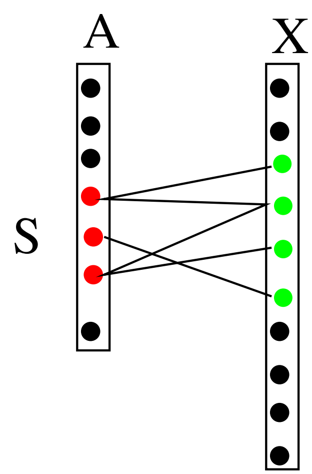
Theorem (PH, Narins, Szabó): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Then $\mathcal{H}$ is a home base hypergraph.

Some proof ingredients

The extremal result for Ryser's conjecture for $r = 3$ initially follows Aharoni's proof of the conjecture for $r = 3$, which uses Hall's Theorem for hypergraphs together with König's Theorem.

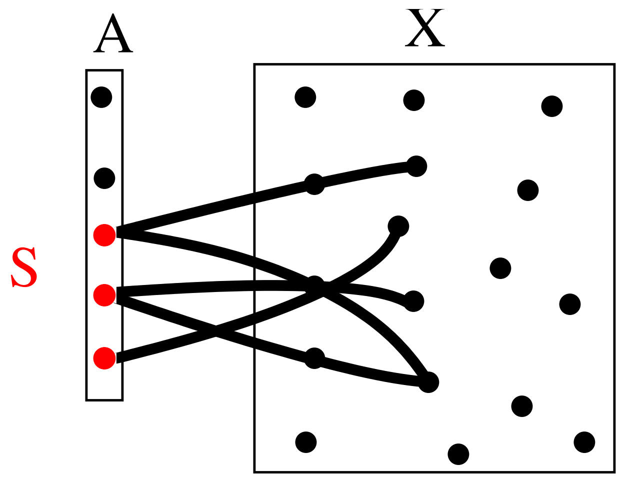
Hall's Theorem: The bipartite graph G has a complete matching if and only if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ is big enough.

Here big enough means $|\Gamma(S)| \geq |S|$.

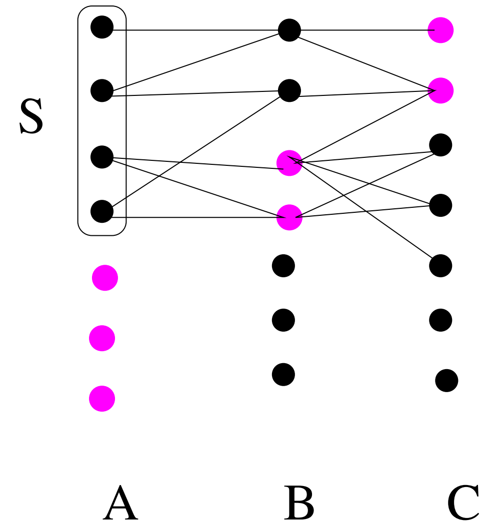


Hall's Theorem for 3-uniform hypergraphs

Theorem (Aharoni, PH, 2000): The bipartite 3-uniform hypergraph H has a complete packing if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ has a matching of size at least $2(|S|-1)+1$.



Aharoni's proof of Ryser for $r = 3$



Let H be a 3-partite 3-uniform hypergraph. Let $\tau = \tau(H)$. Then by König's Theorem, for every subset S of A, the neighbourhood graph $\Gamma(S)$ has a matching of size at least $|S| - (|A| - \tau)$.

Then by a defect version of Hall's Theorem for hypergraphs, we find that H has a packing of size $\lceil \tau/2 \rceil$.

Proof of Hall's Theorem for hypergraphs

The proof has two main steps.

Step 1: The bipartite 3-uniform hypergraph H has a complete packing if: For every subset $S \subseteq A$, the topological connectedness of the matching complex of the neighbourhood graph $\Gamma(S)$ is at least $|S| - 2$.

Step 2: If the graph G has a matching of size at least $2(|S| - 1) + 1$ then the topological connectedness of the matching complex of G is at least $|S| - 2$.

The matching complex of G is the abstract simplicial complex with vertex set $E(G)$, whose simplices are the matchings in G.

Topological connectedness

One way to describe topological connectedness of an abstract simplicial complex Σ, as it is used here:

We say Σ is k-connected if for each $-1 \leq d \leq k$ and each triangulation T of the boundary of a $(d+1)$-simplex, and each function f that labels each point of T with a point of Σ such that the set of labels on each simplex of T forms a simplex of Σ, the triangulation T can be extended to a triangulation T' of the whole $(d+1)$-simplex, and f can be extended to a full labelling f' of T' with the same property.

Hall's Theorem for hypergraphs uses this together with Sperner's Lemma.

The topological connectedness of the matching complex of G is not a monotone parameter.

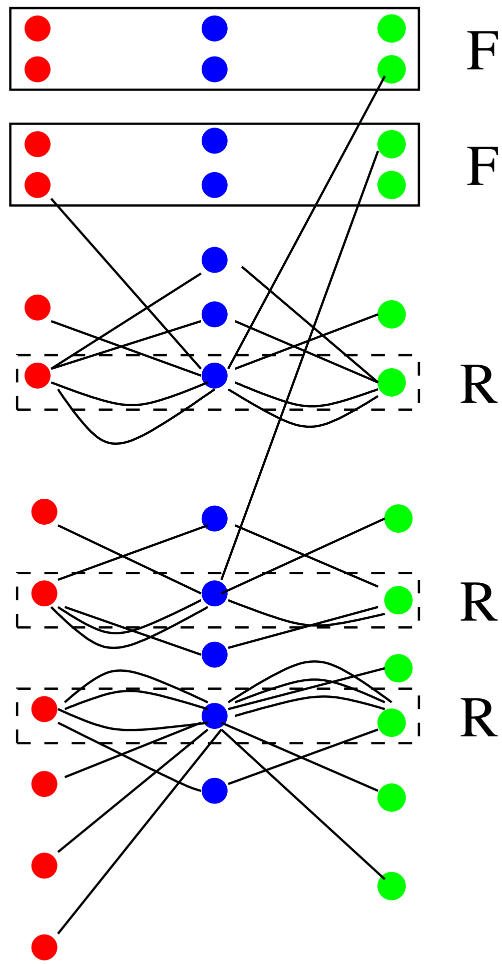
Extremal hypergraphs for Ryser's Conjecture

Two main parts are needed in understanding the extremal hypergraphs for Ryser's Conjecture for $r = 3$.

Part A: Show that any bipartite graph G that has a matching of size $2k$ but whose matching complex has the smallest possible topological connectedness (namely $k - 2$) has a very special structure.

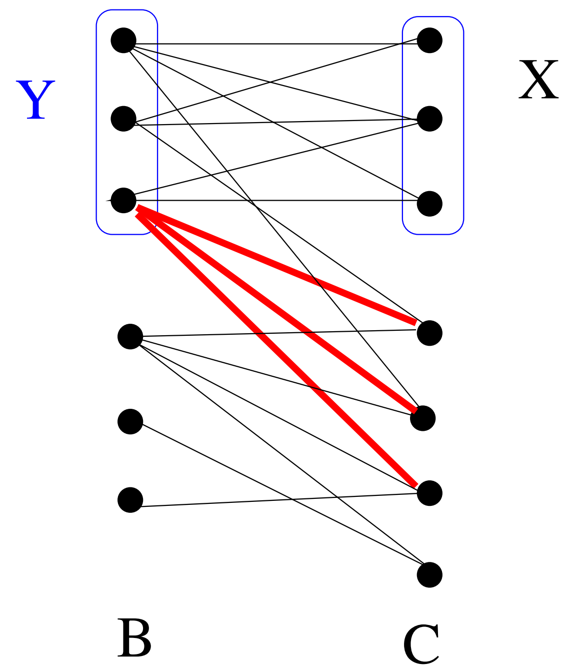
Part B: Analyse how the edges of the neighbourhood graph G of A (which has this special structure) extend to A.

Home base hypergraphs



Part B (one case)

There exists a subset X of C with $|Y| \leq |X|$, where $Y = \Gamma_G(X)$, such that for each $y \in Y$, if we erase the $(y, C \setminus X)$ edges of G, the topological connectedness of the matching complex goes up.



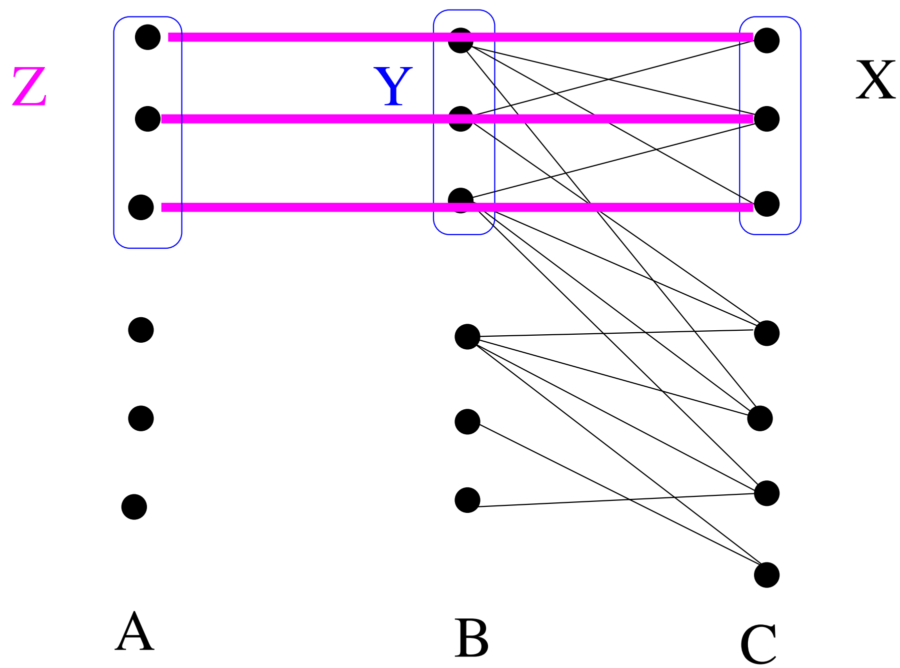
If for each $S \subset A$, the topological connectedness of the matching complex of $\Gamma(S)$ did not go down, then we find H has a packing larger than $\nu(H)$.

So for some S_y, erasing the $(y, C \setminus X)$ edges causes the connectedness to decrease.

Properties of S_y:

- $|S_y| \geq |A| - 1$, which implies $S_y = A \setminus \{a\}$ for some $a \in A$,
- every maximum matching in $\Gamma(S)$ uses an edge of $(y, C \setminus X)$.

What these properties imply



Removing the vertices in Y and Z causes ν to decrease by $|Y|$ and τ to decrease by $2|Y|$. Then we may use induction.